Peking University Summer School Lecture Notes

The 2D Incompressible Boussinesq Equations

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1.1 Introduction

The standard 2D Boussinesq equation can be written as

$$(1.1) \qquad \begin{cases} \vec{u}_t + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \nu \Delta \vec{u} + \theta \vec{e}_2, \\ \nabla \cdot \vec{u} = 0, \\ \theta_t + \vec{u} \cdot \nabla \theta = \kappa \Delta \theta, \end{cases}$$

where $\vec{u}$ denotes the 2D velocity field, p the pressure, θ the temperature in the content of thermal convection and the density in the modeling of geophysical fluids, ν the viscosity, κ the thermal diffusivity, and $\vec{e}_2$ is the unit vector in the vertical direction.

The Boussinesq equations model large scale atmospheric and oceanic flows that are responsible for cold fronts and the jet stream (see the books by A.E. Gill [28], J. Pedlosky [45], A. Majda [38] and others). In addition, the Boussinesq equations also play an important in the study of Rayleigh-Benard convection (see, e.g., the series of papers by P. Constantin and C. Doering [18]).

Mathematically the 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Euler and Navier-Stokes equations such as the vortex stretching mechanism. The inviscid 2D Boussinesq equations are identical to the Euler equations for the 3D axisymmetric swirling flows. The fundamental issue of whether classical solutions to the 3D Euler and Navier-Stokes equations can develop finite time singularities remains outstandingly open and the study of the 2D Boussinesq equations may shed light on this extremely challenging problem.

In order to model anisotropic Boussinesq flows (the viscosity and the thermal diffusivity are different in the horizontal and vertical directions), (1.1) should be written as

$$(1.2) \qquad \begin{cases} u_t + u u_x + v u_y = -p_x + \nu_1 u_{xx} + \nu_2 u_{yy}, \\ v_t + u v_x + v v_y = -p_y + \nu_1 v_{xx} + \nu_2 v_{yy} + \theta, \\ u_x + v_y = 0, \\ \theta_t + u \theta_x + v \theta_y = \kappa_1 \theta_{xx} + \kappa_2 \theta_{yy}, \end{cases}$$

where u and v are the horizontal and vertical components of $\vec{u}$, $\nu_1 \geq 0, \nu_2 \geq 0, \kappa_1 \geq 0$ and $\kappa_2 \geq 0$. (1.2) is called anisotropic Boussinesq equations.

A different type of generalization is to replace the Laplacian by fractional Laplacian, namely

$$(1.3) \qquad \begin{cases} \vec{u}_t + \vec{u} \cdot \nabla \vec{u} = -\nabla p - \nu (-\Delta) \ \vec{u} + \theta \vec{e}_2, \\ \nabla \cdot \vec{u} = 0, \\ \theta_t + \vec{u} \cdot \nabla \theta = -\kappa (-\Delta) \ \theta \end{cases}$$

where $0 < \alpha, \beta \leq 1$. This system will be called fractional Boussinesq equations.

We consider the initial-value problems (IVPs) of the equations in (1.2) and in (1.3) with the initial data

$$\overrightarrow{u}(x,y,0) = \overrightarrow{u}_0(x,y), \qquad \theta(x,y,0) = \theta_0(x,y).$$

Attention is focused on the global regularity issue: Do these IVPs have a global solution for sufficiently smooth data $(\overrightarrow{u}_0, \theta_0)$?

The global regularity problem on the 2D Boussinesq equations has attracted considerable attention recently from the PDE community. Here are some of the people who have worked on it: H. Abidi, D. Adhikari, J.R. Cannon, C. Cao, D. Chae, P. Constantin, R. Danchin, E. DiBenedetto, W. E, T. Hmidi,T. Hou, S. Keraani, S.-K. Kim, A. Larios, C. Li, E. Lunasin, C. Miao, H.-S. Nam, M. Paicu, F. Rousset, C. Shu, E.S. Titi, V. Vicol, J. Wu, X. Xu, L. Xue, to name just a few.

Our lecture will be mainly divided into two large parts: the anisotropic Boussinesq equations and the fractional Boussinesq equations. The part for the anisotropic Boussinesq equations will include the following seven cases: 1) $\nu_1 = \nu_2 = \kappa_1 = \kappa_2 = 0$ (inviscid Boussinesq); 2) $\nu_1 > 0$, $\nu_2 > 0$, $\kappa_1 > 0$ and $\kappa_2 > 0$ (dissipation and thermal diffusion); 3) $\nu_1 > 0$, $\nu_2 > 0$, $\kappa_1 = \kappa_2 = 0$ (dissipation but no thermal diffusion); 4) $\kappa_1 > 0$, $\kappa_2 > 0$, $\nu_1 = \nu_2 = 0$ (thermal diffusion but no dissipation); 5) $\nu_1 > 0$ (horizontal dissipation only); 6) $\kappa_1 > 0$ (horizontal thermal diffusion only); 7) $\nu_2 > 0$ and $\kappa_2 > 0$ (vertical dissipation and vertical thermal diffusion). The recent global regularity result of Cao and Wu [9] on the 2D Boussinesq equations with vertical dissipation will be detailed. The second large part will summarizes recent work on the 2D Boussinesq equations with dissipation given by a fractional Laplacian and presents the work of Chae and Wu on the global regularity of a generalized 2D Boussinesq formulation with a singular velocity [16]. Most of my papers can be found on my website:

http://www.math.okstate.edu/∼jiahong/publications.html

In addition, the book by C. Miao, J. Wu and Z. Zhang [41] would also be very helpful in preparing one for the materials presented here.

1.2 Models

This section explains how the primitive equations, the 2D incompressible Boussinesq equations and the surface quasi-geostrophic (SQG) equation can be formally derived from the 3D rotating Boussinesq equations.

Fluid flows in atmosphere and ocean have two distinctive features: rotation and stratification. The simplest model that contains both features is the 3D rotating Boussinesq equation:

$$\begin{cases} \partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + f\vec{e_3}\times\vec{u} = \nu\Delta\vec{u} - \frac{1}{b}\nabla p + \rho g\,\vec{e_3}, \\ \nabla\cdot\vec{u} = 0, \\ \partial_t\rho + \vec{u}\cdot\nabla\rho = \kappa\Delta\rho, \end{cases} \tag{1.4}$$

where $f = 2\Omega \sin\phi$ with Ω being the angular frequency of planetary rotation and ϕ the latitude, ρ_b is a constant for reference density, $\vec{e_3} = (0,0,1)$, $f\vec{e_3} \times \vec{u}$ represents the Coriolis forcing. More explicitly, if $\vec{u} = (u, v, w)$, then

$$f\vec{e_3} \times \vec{u} = f \begin{pmatrix} -v \\ u \\ 0 \end{pmatrix}$$

and (1.4) becomes

$$(1.5) \qquad \begin{cases} \partial_t u + u\partial_x u + v\partial_y u + w\partial_z u - fv = \nu\Delta u - \frac{1}{{}_b}\partial_x p, \\ \partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + fu = \nu\Delta v - \frac{1}{{}_b}\partial_y p, \\ \partial_t w + u\partial_x w + v\partial_y w + w\partial_z w = \nu\Delta w - \frac{1}{{}_b}\partial_z p + \rho g, \\ \partial_x u + \partial_y v + \partial_z w = 0, \\ \partial_t \rho + u\partial_x \rho + v\partial_y \rho + w\partial_z \rho = \kappa\Delta\rho. \end{cases}$$

For atmospheric and oceanic flows in the mid-latitude, the $w-$equation can be simplified. The terms involving w in the $w-$equation are small and the w-equation is reduced to

$$(1.6) \qquad \frac{1}{\rho_b}\partial_z p - \rho g = 0.$$

This is the so called the hydrostatic balance. (1.6) provides a special solution of (1.5). In fact, $\vec{u} = 0$ with p and ρ satisfying (1.6) solves (1.5). The system of equations containing the u-equation, the v-equation in (1.5) and

$$\begin{aligned} &\frac{1}{\rho_b}\partial_z p - \rho g = 0, \\ &\partial_x u + \partial_y v + \partial_z w = 0, \\ &\partial_t \rho + u\partial_x \rho + v\partial_y \rho + w\partial_z \rho = \kappa\Delta\rho \end{aligned}$$

are called the primitive equations. The primitive equations have been studied by many authors (see, e.g. [37], [7]).

If $f \equiv 0$, (1.5) becomes the 3D Boussinesq equations without rotation. If $f \equiv 0$ and all physical quantities are independent of z, then (1.5) reduces to the 2D Boussinesq equations, which read

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = -\partial_x p + \nu\Delta u, \\ \partial_t v + u\partial_x v + v\partial_y v = -\partial_y p + \nu\Delta v + \theta, \\ \partial_x u + \partial_y v = 0, \\ \partial_t \theta + u\partial_x \theta + v\partial_y \theta = \kappa\Delta\theta. \end{cases}$$

Under some circumstances, the 3D rotation Boussinesq reduces to the surface quasi-geostrophic (SQG) equation. The Rossby number indicates the ratio of the inertial to the strength of rotation. In low-pressure systems, the Rossby number is small and the balance is between Coriolis and pressure forces, namely

$$f\vec{e_3} \times \vec{u} = -\frac{1}{\rho_b}\nabla p.$$

This is the so-called geostrophic balance. In terms of their components,

$$f\begin{pmatrix} -v \\ u \end{pmatrix} = -\frac{1}{\rho_b}\begin{pmatrix} \partial_x p \\ \partial_y p \end{pmatrix}$$

or simply $f\,\rho_b\,\vec{u}_H = \nabla^\perp p$. After ignoring the dissipation and removing the geostrophic balance from (1.5),

$$\partial_t u + u\partial_x u + v\partial_y u = 0,$$
$$\partial_t v + u\partial_x v + v\partial_y v = 0.$$

Then, $\omega = \partial_x v - \partial_y u$ satisfies

$$\partial_t \omega + u\partial_x \omega + v\partial_y \omega = 0, \quad \frac{D\omega}{Dt} = 0. \tag{1.7}$$

In terms of the stream function ψ,

$$\omega = \Delta\psi = \partial_x^2\psi + \partial_y^2\psi + \frac{\partial}{\partial z}\left(\left(\frac{f}{N}\right)^2 \frac{d\psi}{dz}\right)$$

where $N = \sqrt{-g\partial_z \bar{\rho}}$ denotes the the buoyancy frequency. (1.7) indicates that, if ω is a constant initially, it remains a constant. Suppose that $\omega_0 = 0$ and re-scaling the $z-$component, we find that

$$\Delta\psi = 0.$$

In addition, for small Rossby number and through re-scaling,

$$\rho = \frac{\partial\psi}{\partial z}$$

and

$$\partial_t \rho + u\partial_x \rho + v\partial_y \rho = 0 \quad \text{or} \quad \frac{D\rho}{Dt} = 0.$$

Lemma 1.2.1. *If g is a bounded smooth function in $\mathbb{R}^d$, then*

$$\begin{cases} \Delta\psi = 0 & \mathbb{R}^d \times \mathbb{R}^+ \\ \psi = g & \mathbb{R}^d \end{cases}$$

has a bounded smooth solution ψ, and

$$\left.\frac{\partial\psi}{\partial z}\right|_{z=0} = (-\Delta)^{\frac{1}{2}} g \quad on \quad \mathbb{R}^d$$

Applying this lemma and writing θ for ρ, we obtain

$$\begin{cases} \partial_t \theta + u\partial_x \theta + v\partial_y \theta = 0 \\ u = \nabla^\perp \psi, \quad (-\Delta)^{\frac{1}{2}}\psi = \theta, \end{cases}$$

which is the SQG equation.

1.3 The inviscid 2D Boussinesq equations

This section provides some existing results on the global regularity problem concerning the inviscid 2D Boussinesq equations. Recall that the inviscid 2D Boussinesq equations are given by

$$\begin{cases} \partial_t \vec{u} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\vec{\nabla} p + \theta \vec{e}_2, \\ \nabla \cdot \vec{u} = 0, \\ \partial_t \theta + (\vec{u} \cdot \vec{\nabla})\theta = 0, \end{cases} \tag{1.8}$$

where $\vec{e}_2$ denotes the unit vector in the vertical direction. The issue is whether or not (1.8) has a global solution when supplemented with a reasonably "regular" data,

$$\vec{u}(x,0) = \vec{u}_0(x), \quad \theta(x,0) = \theta_0(x).$$

The global regularity problem for a general data is currently open. In fact, the inviscid Boussinesq can be identified with the 3D axisymmetric Euler with swirl. Recall the 3D axisymmetric Euler equations

$$u_t + (u^r \partial_r + u^z \partial_z) u \ + \frac{u^r u}{r} = 0, \qquad \frac{\widetilde{D}}{Dt}(ru \ ) = 0, \quad \frac{\widetilde{D}}{Dt}(ru \ )^2 = 0$$

where $\frac{\widetilde{D}}{Dt} = \partial_t + (u^r \partial_r + u^z \partial_z)$. The swirl component of the vorticity ω satisfies

$$\frac{\widetilde{D}}{Dt}\left(\frac{\omega}{r}\right) = -\frac{1}{r^4}\partial_z (ru \ )^2$$

While the 2D Boussinesq is given by, with $\frac{D}{Dt} = \partial_t + u \cdot \nabla$

$$\frac{D}{Dt}\theta = 0, \qquad \frac{D}{Dt}\omega = -\partial_{x_1}\theta.$$

We do have local well-posedness and several regularity criteria (see, e.g., W. E and C. Shu [27], D. Chae and H. Nam [15], Chae, S. Kim and H. Nam [14], R. Danchin [22]).

Theorem 1.3.1. *Assume $u_0 \in H^s(\mathbb{R}^2)$ and $\theta_0 \in H^s(\mathbb{R}^2)$ with $s > 2$. Then there is $T = T(||(u_0, \theta_0)||_{H^s}) > 0$ such that the 2D inviscid Boussinesq has a unique solution $(u, \theta) \in C([0,T]; H^s)$. Furthermore, if*

$$\int_0^{T^*} ||\nabla u||_{L^\infty} \, dt < +\infty \quad or \quad \int_0^{T^*} ||\nabla \theta||_{L^\infty} \, dt < +\infty,$$

then the local solution can be extended to $[0, T^*]$.

The local well-posedness can be easily established through the standard ODE theory such as the contraction mapping principle or successive approximation. We remark that

$s > 2$ appears to be necessary and it is not clear whether or not the local well-posedness can be obtained in the borderline space H^2. The main reason is that H^1 is not necessarily embedded in L^∞. A recent work of Chae and Wu partially addressed this issue (see [17]). In the following we provide the proof for the regularity criteria and start with several simple facts.

The first fact is a commutator estimate (see, e.g., Kenig, Ponce and Vega [34]). For any $s \in \mathbb{R}$, we write

$$J^s f = (1-\Delta)^{\frac{s}{2}} f \quad \text{or} \quad \widehat{J^s f(\xi)} = (1+|\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi).$$

Lemma 1.3.1. *Let $s > 0$ and $1 < p < \infty$. Then*

$$\begin{aligned} &\|J^s(fg)\|_{L^p} \le C\left(\|f\|_{W^{s,p_1}} \|g\|_{L^{p_2}} + \|g\|_{W^{s,p_3}} \|f\|_{L^{p_4}}\right), \\ &\|J^s(fg) - f J^s g\|_{L^p} \le C\left(\|f\|_{W^{s,p_1}} \|g\|_{L^{p_2}} + \|g\|_{W^{s-1,p_3}} \|\nabla f\|_{L^{p_4}}\right), \end{aligned}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$ with $p_2, p_4 \in [1,\infty]$ and $p_1, p_3 \in (1,\infty)$, and C's are constants depending on s, p, p_1, p_2, p_3 and p_4. In addition, these inequalities remain valid when $J^s f$ is replaced by $\Lambda^s f$, where

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \widehat{f}(\xi).$$

The next lemma bounds the L^q-norm of the solution to a simple transport equation.

Lemma 1.3.2. *Let $T > 0$ and let $q \in [1,\infty]$. Assume that $\omega_0 \in L^q(\mathbb{R}^d)$ and $f \in L^1([0,T]; L^q(\mathbb{R}^d))$* f ‖ [illegible]

Proof of Lemma 1.3.3. This lemma can be easily proven via the Littlewood-Paley decomposition and Besov space techniques. Writing

$$\nabla u = \sum_{j=-1}^{\infty} \Delta_j \nabla u,$$

we have, by the fact that $\|\Delta_j \nabla u\|_{L^\infty} \leq C\|\Delta_j \omega\|_{L^\infty}$ for any $j \geq 0$ and Bernstein's inequality,

$$\begin{aligned}
\|\nabla u\|_{L^\infty} &\leq \|\Delta_{-1}\nabla u\|_{L^\infty} + \sum_{j=0}^{N} C\|\Delta_j \omega\|_{L^\infty} + \sum_{j=N+1}^{\infty} 2^j \|\Delta_j \omega\|_{L^2} \\
&\leq C\|u\|_{L^2} + C(N+1)\|\omega\|_{L^\infty} + \sum_{j=N+1}^{\infty} 2^{j(1-\ )} 2^{\ j} \|\Delta_j \omega\|_{L^2} \\
&\leq C\|u\|_{L^2} + C(N+1)\|\omega\|_{L^\infty} + \|\omega\|_{H^\sigma} \left(\sum_{j=N+1}^{\infty} 2^{2j(1-\ )} \right)^{\frac{1}{2}} \\
&\leq C\|u\|_{L^2} + C(N+1)\|\omega\|_{L^\infty} + \|\omega\|_{H^\sigma} 2^{(N+1)(1-\ )} C(\sigma)
\end{aligned}$$

Now taking an integer N such that $\|\omega\|_{H^\sigma} 2^{(N+1)(1-\ )} \leq 1$, we then have $N+1 \leq \frac{1}{\ -1} \log_2(1 + \|\omega\|_{H^\sigma})$. Therefore

$$\|\nabla u\|_{L^\infty} \leq C(1 + \|u\|_{L^2}) + C\|\omega\|_{L^\infty} \log(1 + \|\omega\|_{H^\sigma}).$$

This completes the proof of Lemma 1.3.3. □

Proof of theorem 1.3.1. The local existence and uniqueness can be obtained by the Picard type theorems. One can follow the steps leading to the local well-posedness result for the Euler and the Navier-Stokes equations, as in the book of Majda and Bertozzi [39].

Let (u, θ) be a local solution and assume that $\int_0^{T^*} \|\nabla u\|_{L^\infty} dt < \infty$. The goal is to obtain a priori bound for H^s-norm on $[0, T^*]$. To do so, we start with the vorticity formulation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \partial_x \theta, \\ \partial_t \theta + u \cdot \nabla \theta = 0. \end{cases}$$

Applying J^{s-1} to the vorticity equation, multiplying by $J^{s-1}\omega$ and integrating in space, we obtain, after integration by parts,

$$\frac{1}{2}\frac{d}{dt}\|J^{s-1}\omega\|_{L^2}^2 = \int \partial_x J^{s-1}\theta J^{s-1}\omega - \int J^{s-1}\omega (J^{s-1}\nabla \cdot (u\omega) - u \cdot J^{s-1}\nabla\omega).$$

By Lemma 1.3.1,

$$\|J^{s-1}\nabla \cdot (u\omega) - u \cdot J^{s-1}\nabla\omega\|_{L^2} \leq C \left(\|J^s u\|_{L^2} \|\omega\|_{L^\infty} + \|J^{s-1}\omega\|_{L^2} \|\nabla u\|_{L^\infty} \right).$$

Therefore, by the simple fact $\|J^s u\|_{L^2} = \|J^{s-1}\omega\|_{L^2}$,

$$\frac{d}{dt}\|J^{s-1}\omega\|_{L^2}^2 \leq \|J^s\theta\|_{L^2}^2 + \|J^{s-1}\omega\|_{L^2}^2 + C\,\|J^{s-1}\omega\|_{L^2}^2\|\nabla u\|_{L^\infty}$$

Similarly

$$\frac{d}{dt}\|J^s\theta\|_{L^2}^2 \leq C\,\|J^s\theta\|_{L^2}(\|J^s u\|_{L^2}\|\nabla\theta\|_{L^\infty} + \|J^s\theta\|_{L^2}\|\nabla u\|_{L^\infty}).$$

Furthermore, it following from $\partial_t\theta + u\cdot\nabla\theta = 0$ that, for any $q \in [1,\infty]$,

$$\|\nabla\theta\|_{L^q} \leq \|\nabla\theta_0\|_{L^q} e^{\int_0^t \|\nabla u\|_{L^\infty} d}\ .$$

In fact, by a standard energy estimate, we have, for any $1 \leq q < \infty$,

$$\frac{1}{q}\frac{d}{dt}\|\nabla\theta\|_{L^q}^q \leq \|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^q}^q,$$

which leads to the desired inequality for $q \in (1,\infty)$ and the case $q = \infty$ is obtained by taking $q \to \infty$. Therefore, $Y(t) = \|J^{s-1}\omega\|_{L^2}^2 + \|J^s\theta\|_{L^2}^2$ obeys

$$\frac{d}{dt}Y(t) \leq \|\nabla\theta_0\|_{L^\infty} e^{2\int_0^t \|\nabla u\|_{L^\infty} d}\ Y(t) + C\,Y(t)(1 + \|\nabla u\|_{L^\infty})$$

By Gronwall's inequality, if $\int_0^{T^*} \|\nabla u\|_{L^\infty} dt < \infty$ then $Y(t) < \infty$ on $[0, T^*]$ or $(u,\theta) \in H^s$.

Now we show that $\int_0^{T^*} \|\nabla\theta\|_{L^\infty} dt < +\infty$ implies that $(u,\theta) \in H^s$ for $t \in [0, T^*]$. According to the estimates above

$$\frac{d}{dt}Y(t) \leq CY(t)(\|\nabla\theta\|_{L^\infty} + 1 + \|\nabla u\|_{L^\infty}).$$

By $\partial_t\omega + u\cdot\nabla\omega = \partial_x\theta$ and Lemma 1.3.2, we have

$$\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + \int_0^t \|\nabla\theta\|_{L^\infty} d\tau.$$

By Lemma 1.3.3, for $\sigma > 1$,

$$\|\nabla u\|_{L^\infty} \leq C(1 + \|u\|_{L^2}) + C\,\|\omega\|_{L^\infty}\log(1 + \|\omega\|_{H^\sigma}).$$

Therefore, for $s > 2$,

$$\begin{aligned}\frac{dY(t)}{dt} &\leq CY(t)(1 + \|\nabla\theta\|_{L^\infty} + \|u\|_{L^2} + \|\omega\|_{L^\infty}\log(1 + \|J^{s-1}\omega\|_{L^2}))\\ &\leq CY(t)(1 + \|\nabla\theta\|_{L^\infty} + \|u\|_{L^2} + \|\omega\|_{L^\infty}\log(1 + Y(t))),\end{aligned}$$

which implies immediately that, if $\int_0^{T^*} \|\nabla\theta\|_{L^\infty} dt < \infty$, then $Y(t) < \infty$ on $[0, T^*]$. this completes the proof of Theorem 1.3.1. □

Finally we make a remark.

Remark 1.3.2. *For the 3D Euler equation, $u_t + u \cdot \nabla u = -\nabla p,\ \nabla \cdot u = 0$, we have the Beale-Kato-Majda criterion, which says that if*

$$\int_0^T \|\omega(\cdot,t)\|_\infty \, dt < \infty, \tag{1.9}$$

then the solution remains regular on $[0,T]$. However, for the inviscid 2D Boussinesq equations, it is unknown if

$$\int_0^T \|\omega(\cdot,t)\|_\infty \, dt < \infty$$

is enough for the global regularity. This does not appear to be trivial.

What is the difficulty? Following the standard idea, we try the energy estimate. Applying $J^s = (I-\Delta)^{\frac{s}{2}}$ to $\partial_t \theta + u \cdot \nabla \theta = 0$,

$$\partial_t J^s \theta + u \cdot \nabla J^s \theta = -J^s(u \cdot \nabla \theta) + u \cdot \nabla J^s \theta$$

$$\frac{1}{2}\frac{d}{dt}\|J^s\theta\|_{L^2}^2 = \int (-J^s(u \cdot \nabla \theta) + u \cdot \nabla J^s \theta) J^s \theta$$

By the commutator estimate,

$$\| - J^s(u \cdot \nabla \theta) + u \cdot \nabla J^s \theta\|_{L^2} \le C(\|\nabla u\|_{L^\infty}\|\theta\|_{H^s} + \|\nabla \theta\|_{L^\infty}\|u\|_{H^s}).$$

Then,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{H^s}^2 \le C\|\nabla u\|_{L^\infty}\|\theta\|_{H^s}^2 + \|\nabla \theta\|_{L^\infty}\|u\|_{H^s}\|\theta\|_{H^s}.$$

If $\|\nabla\theta\|_{L^\infty}$ were not there, then we would combine

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 \le C\|\nabla u\|_{L^\infty}\|u\|_{H^s}^2 + \|u\|_{H^s}\|\theta\|_{H^s}$$

with the logarithmic interpolation inequality

$$\|\nabla u\|_{L^\infty} \le C\|u\|_{L^2} + C\|\omega\|_{L^\infty}\log(1 + \|u\|_{H^s})$$

to obtain an ODE of the form, $X = \|u\|_{H^s}^2 + \|\theta\|_{H^s}^2$,

$$\frac{d}{dt}X(t) \le C\|\omega\|_{L^\infty} X(t)\log(1 + X(t)) + C\|u\|_{L^2}.$$

But unfortunately it appears to be difficult to control $\|\nabla\theta\|_{L^\infty}$ in terms of the vorticity alone.

1.4 The viscous 2D Boussinesq equations

This section deals with the 2D viscous Boussinesq equations,

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + \theta \vec{e_2}, \\ \nabla \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta, \end{cases} \tag{1.10}$$

where both ν and κ are positive numbers. The global regularity can be established for this system of equations (see, e.g., [6]).

Theorem 1.4.1. *Given an initial data $(u_0, \theta_0) \in H^2(\mathbb{R}^2)$. The 2D viscous Boussinesq equations (1.10) have a unique global classical solution $(u, \theta) \in C([0, \infty), H^2)$.*

Proof. The proof of this result is almost trivial and similar to that for the 2D Navier-Stokes equations. The proof is provided here to point out why the global regularity problem for the 2D problem can be easily solved and why the 3D case is different. It suffices to establish the global H^1 bound. First of all, we have the L^2-bounds

$$\|\theta\|_2^2 + 2\kappa \int_0^t \|\nabla \theta\|_2^2 d\tau = \|\theta_0\|_2^2,$$

$$\|u\|_2^2 + 2\nu \int_0^t \|\nabla u\|_2^2 d\tau \leq (\|u_0\|_2 + t\|\theta_0\|_2)^2.$$

It follows from the vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega + \partial_{x_1} \theta$$

that

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \nu \|\nabla \omega\|_2^2 \leq \|\theta\|_2^2 \|\partial_{x_1}\omega\|_2^2 \leq \frac{\nu}{2}\|\nabla\omega\|_2^2 + \frac{1}{2\nu}\|\theta_0\|_2^2$$

$$\|\omega\|_2^2 + \nu \int_0^t \|\nabla \omega\|_2^2 d\tau \leq \|\omega_0\|_2^2 + \frac{1}{\nu} t \|\theta_0\|_2^2$$

By the equation for θ,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_2^2 + \kappa\|\Delta\theta\|_2^2 \leq \int |\nabla u||\nabla\theta|^2 dx \leq C\,\|\nabla u\|_3 \|\nabla\theta\|_3^2. \tag{1.11}$$

Applying the Gagliardo-Nirenberg inequality

$$\|f\|_{L^3(\mathbb{R}^d)} \leq C\,\|f\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{6}}\,\|\nabla f\|_{L^2(\mathbb{R}^d)}^{\frac{d}{6}}, \tag{1.12}$$

with $d = 2$, namely $\|f\|_3 \leq C\,\|f\|_2^{2=3}\,\|\nabla f\|_2^{1=3}$, we obtain

$$\|\nabla u\|_3^3 \leq C\,\|\nabla u\|_2^2\,\|\nabla^2 u\|_2 \leq \frac{\nu}{2}\|\nabla^2 u\|_2^2 + \frac{C}{\nu}\|\nabla u\|_2^2\,\|\nabla u\|_2^2$$

Therefore,

$$C\,\|\nabla u\|_3\|\nabla\theta\|_3^2 \le \frac{\nu}{2}\|\nabla^2 u\|_2^2 + \frac{\kappa}{2}\|\nabla^2\theta\|_2^2 + \frac{C}{\nu}\|\nabla u\|_2^2\,\|\nabla u\|_2^2 + \frac{C}{\kappa}\|\nabla\theta\|_2^2\,\|\nabla\theta\|_2^2.$$

Inserting this inequality in (1.11) and applying the integrability

$$\int_0^\infty \|\nabla u\|_2^2\,d\tau < \infty, \quad \int_0^\infty \|\nabla\theta\|_2^2\,d\tau < \infty,$$

we have, for any $T > 0$,

$$\|\nabla\theta(T)\|_2^2 + \kappa\int_0^T \|\Delta\theta\|_2^2 d\tau \le C(T).$$

H^2 norm can be obtained through a similar procedure. We would like to point out that (1.12) depends on the dimension d and a similar procedure does not yield a global H^1 bound in the 3D case. When $d = 3$, we have

$$\|\nabla u\|_{L^3(\mathbb{R}^3)}^3 \le C\,\|\nabla u\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}}\,\|\nabla^2 u\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} \le \frac{\nu}{2}\|\nabla^2 u\|_{L^2(\mathbb{R}^3)}^2 + \frac{C}{\nu}\|\nabla u\|_{L^2(\mathbb{R}^3)}^4\,\|\nabla u\|_{L^2(\mathbb{R}^3)}^2.$$

But now $\|\nabla u\|_{L^2(\mathbb{R}^3)}^4$ is no longer time integrable. This completes the proof of Theorem 1.4.1. □

We finally remark that the solutions to (1.10) in the basic energy space $(u,\theta) \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$ can be shown to be unique.

1.5 Partial dissipation with either Δu or $\Delta\theta$

This section studies the global regularity of the 2D Boussinesq equations with partial dissipation. We start with the case when only Δu is present.

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + \nu\Delta u + \theta\vec{e}_2, \\ \nabla\cdot u = 0, \\ \partial_t\theta + u\cdot\nabla\theta = 0. \end{cases} \tag{1.13}$$

The global regularity issue for this system of equations was listed by H.K. Moffatt as one of the 21st century PDE problems [44]. The global regularity was obtained by T. Hou and C. Li [33] and by D. Chae [10]. The result can be stated as follows.

Theorem 1.5.1. *Let* $(u_0,\theta_0) \in W^{2;q}(\mathbb{R}^2)$ *with* $2 < q < \infty$. *Then (1.13) has a unique global solution* $(u,\theta) \in C([0,\infty); W^{2;q}(\mathbb{R}^2))$.

Proof. First of all, for any $q \in [1,\infty]$, $\|\theta(t)\|_q \le \|\theta_0\|_q$ and

$$\|u\|_2^2 + \nu \int_0^t \|\nabla u\|_2^2 d\tau \le (\|u_0\|_2 + t\|\theta_0\|_2)^2.$$

In particular, due to $\|\nabla u\|_{L^2} = \|\omega\|_{L^2}$, for any $t > 0$,

$$\nu \int_0^t \|\omega\|_{L^2}^2 \, dt < \infty.$$

For $2 \le q < \infty$, it follows from $\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega + \partial_{x_1}\theta$ that

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\|\omega\|_q^q + (q-1)\nu \int |\nabla\omega|^2 |\omega|^{q-2} dx \\
= \ & \int \partial_{x_1}\theta \omega |\omega|^{q-2} dx \\
= \ & (q-1)\int \theta \partial_{x_1}\omega |\omega|^{q-2} \\
= \ & (q-1)\int \theta |\omega|^{\frac{q-2}{2}} \partial_{x_1}\omega |\omega|^{\frac{q-2}{2}} \\
\le \ & (q-1)\|\nabla\omega|\omega|^{\frac{q-2}{2}}\|_2 \|\theta\|_q \|\omega\|_q^{\frac{q-2}{2}} \\
\le \ & \frac{\nu(q-1)}{2}\int |\nabla\omega|^2|\omega|^{q-2}dx + (q-1)\frac{1}{2\nu}\|\theta\|_q^2\|\omega\|_q^{q-2}.
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|\omega\|_q^q + q(q-1)\nu \int |\nabla\omega|^2|\omega|^{q-2} \le \frac{q(q-1)}{\nu}\|\theta\|_q^2\|\omega\|_q^{q-2}.$$

In particular,

$$\frac{d}{dt}\|\omega\|_q^2 \le \frac{2(q-1)}{\nu}\|\theta_0\|_q^2$$

$$\|\omega\|_q^2 \le \|\omega_0\|_q^2 + \frac{2(q-1)}{\nu} t \|\theta_0\|_q^2$$

$$\|\omega\|_q \le \|\omega_0\|_q + \sqrt{\frac{2}{\nu} t (q-1)} \|\theta_0\|_q. \tag{1.14}$$

We thus have obtained a global (in time) bound for $\|\omega\|_q$ with any $2 \le q < \infty$, but this does not lead to a global bound for $\|\omega\|_\infty$. As a consequence,

$$\|u\|_\infty \le C\, \|u\|_2^a \|\nabla u\|_q^{1-a} \le C\, \|u\|_2^a \|\omega\|_q^{1-a} < \infty,$$

where $q \in (2,\infty)$ and $a = \frac{q-2}{2(q-1)}$. We now bound $\|\nabla\omega\|_q$ and $\|\nabla\theta\|_q$.

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\|\nabla\omega\|_q^q + \nu \int |\Delta\omega|^2 |\nabla\omega|^{q-2} + \frac{1}{4}(q-2)\nu \int |\nabla|\nabla\omega|^2|^2 |\nabla\omega|^{q-4} \\
&\qquad = -\int \nabla(u\cdot\nabla\omega)\nabla\omega|\nabla\omega|^{q-2} + \int \partial_{x_1}\nabla\theta \cdot \nabla\omega |\nabla\omega|^{q-2}
\end{aligned} \tag{1.15}$$

To see why the dissipation yields the second and third term on the left-hand side, we write the inner product in terms of the components and integrate by parts

$$\begin{aligned}
&\int \nabla\Delta\omega\cdot\nabla\omega|\nabla\omega|^{q-2}\\
=\;&\int \partial_j\partial_k\partial_k\omega\partial_j\omega|\nabla\omega|^{q-2}\\
=\;&-\int(\partial_j\partial_k\omega)^2\,|\nabla\omega|^{q-2}-(q-2)\int(\partial_j\partial_k\omega\partial_j\omega)^2|\nabla\omega|^{q-4},
\end{aligned}$$

where repeated indices are summed. To estimate the terms on the right, we integrate by parts,

$$\begin{aligned}
&\int u\cdot\nabla\omega\nabla(\nabla\omega|\nabla\omega|^{q-2})\\
=\;&\int u\cdot\nabla\omega\left(\Delta\omega|\nabla\omega|^{q-2}+\frac{q-2}{2}\nabla\omega\nabla(|\nabla\omega|^2)|\nabla\omega|^{q-4}\right)\\
\le\;&\|u\|_\infty\|\nabla\omega\|_q\|\nabla\omega\|_q^{\frac{q}{2}-1}\|\Delta\omega|\nabla\omega|^{\frac{q-2}{2}}\|_2\\
&+\|u\|_\infty\|\nabla\omega\|_q\frac{q-2}{2}\|\nabla\omega\|_q^{\frac{q-2}{2}}\|\nabla|\nabla\omega|^2|\nabla\omega|^{\frac{q-4}{2}}\|_2.
\end{aligned}$$

Then, the entire right hand side of (1.15) can be bounded by

$$\frac{\nu}{2}\|\Delta\omega|\nabla\omega|^{\frac{q-2}{2}}\|_2^2+\frac{4(q-1)}{\nu}(\|u\|_\infty^2\|\nabla\omega\|_q^q+\|\nabla\theta\|_q^q)+\frac{q-2}{2}\nu\|\nabla(|\nabla\omega|^2)|\nabla\omega|^{\frac{q-4}{2}}\|_2^2.$$

For the θ equation, we can get

$$\frac{1}{q}\frac{d}{dt}\|\nabla\theta\|_q^q\le\|\nabla u\|_\infty\|\nabla\theta\|_q^q.$$

Then

$$\frac{1}{q}\frac{d}{dt}(\|\nabla\omega\|_q^q+\|\nabla\theta\|_q^q)\le\frac{4}{\nu}(q-1)\|u\|_\infty^2\|\nabla\omega\|_q^q+(\frac{4}{\nu}(q-1)+\|\nabla u\|_\infty)\|\nabla\theta\|_q^q$$

We don't know if $\int_0^t\|\nabla u\|_\infty dt<\infty$. We need a logarithmic type inequality. By lemma 1.5.1 below,

$$\|\nabla u\|_{L^\infty}\le c(1+\|w\|_{H^1})\left[\log(1+\|\nabla w\|_{L^q})\right]^{\frac{1}{2}}.$$

It then follows that $Y(t)=\|\nabla w\|_{L^q}^q+\|\nabla\theta\|_{L^q}^q$ satisfies

$$\frac{1}{q}\frac{dY(t)}{dt}\le cY(t)+c(1+\|w\|_{H^1})Y(t)\left[\log(1+Y(t))\right]^{\frac{1}{2}}.$$

Since, for any $T>0$,

$$\int_0^T\|w\|_{H^1}\,dt<C(T),$$

we have

$$Y(t) \leq C(T).$$

Together with the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{L^\infty} \leq c \|\nabla u\|_{L^2}^{a} \|\nabla w\|_{L^q}^{1-a} \quad \text{with} \quad a =$$

Now taking an integer N such that

$$\|f\|_{\dot{B}^{d/2}_{q,\infty}} 2^{(N+1)d(\frac{1}{q}-\frac{1}{2})} \le 1,$$

we then obtain (1.16). The inequality in (1.17) can be proven by slightly changing the proof above. In fact, if we employ the following bound instead of the one above

$$\sum_{j=0}^{N} 2^{jd(\frac{1}{2}-\frac{1}{\infty})}\|\triangle_j f\|_{L^2} \le (N+1)\sup_j 2^{jd(\frac{1}{2}-\frac{1}{\infty})}\|\triangle_j f\|_{L^2} = (N+1)\|f\|_{\dot{B}^{d/2}_{2,\infty}},$$

then the procedure above leads to (1.17). This completes the proof of Lemma 1.5.1.

Next, we consider the system with thermal diffusion $\Delta\theta$, namely

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = -\nabla p + \theta\overrightarrow{e_2} \\ \partial_t\theta + (u\cdot\nabla)\theta = \kappa\Delta\theta \end{cases} \tag{1.18}$$

with $\kappa > 0$. This system of equations has also been shown to possess global solutions for sufficiently smooth data. More precisely, we have the following theorem,

Theorem 1.5.2. *Let $(\theta_0, u_0) \in W^{2;q}(\mathbb{R}^2), 2 < q < \infty$. Then the system has a unique global solution*

$$(u,\theta) \in C([0,\infty); W^{2;q}).$$

The following theorem is proven in a very similar way.

Proof. The local existence and uniqueness part is standard. To obtain the global estimate, we reply more on the regularity of θ due to the diffusion in the θ-equation. Clearly, we have the basic bounds, for any $q \in [2,\infty)$,

$$\|\theta\|^q_{L^q} + q(q-1)\kappa\int_0^t\int |\nabla\theta|^2|\theta|^{q-2}dxd\tau = \|\theta_0\|^q_{L^q}$$

$$\|u\|_{L^2} \le \|u_0\|_{L^2} + t\|\theta_0\|_{L^2}$$

From the vorticity equation $\partial_t\omega + (u\cdot\nabla)\omega = \partial_{x_1}\theta$, we get

$$\frac{1}{2}\partial_t\|\omega\|^2_{L^2} = \int(\partial_{x_1}\theta)\omega \le \|\partial_{x_1}\theta\|_{L^2}\|\omega\|_{L^2}$$

Therefore,

$$\|\omega(T)\|_{L^2} \le \|w_0\|_{L^2} + \int_0^T \|\partial_{x_1}\theta\|_{L^2}dt \le \|\omega_0\|_{L^2} + \left(\int_0^T \|\nabla\theta\|^2_{L^2}dt\right)^{\frac{1}{2}}\sqrt{T}.$$

Next,

$$\frac{1}{q}\partial_t\|w\|^q_{L^q} = \int(\partial_{x_1}\theta)w|w|^{q-2}dx \le \|\nabla\theta\|_{L^q}\|w\|^{q-1}_{L^q}.$$

Or somply

$$\partial_t \|w\|_{L^q} \leq c\|\nabla\theta\|_{L^q}.$$

On the other hand,

$$\begin{aligned}
& \frac{1}{q}\partial_t \|\nabla\theta\|_{L^q}^q + \kappa \int |\Delta\theta|^2 |\nabla\theta|^{q-2} + \kappa(\frac{q-2}{4}) \int |\nabla|\nabla\theta|^2|^2 |\nabla\theta|^{q-4} dx \\
= & -\int \nabla((u\cdot\nabla)\theta)\cdot\nabla\theta|\nabla\theta|^{q-2} \\
\leq & \; C\,\|u\|_{L^\infty} \left[(q-1)\|\nabla\theta\|_{L^q}^{\frac{q}{2}} \|\nabla|\nabla\theta|^2|\nabla\theta|^{\frac{q-4}{2}}\|_{L^2} + \|\nabla\theta\|_{L^q}^{\frac{q}{2}} \|\Delta\theta|\nabla\theta|^{\frac{q-2}{q}}\|_{L^2}\right].
\end{aligned}$$

Then

$$\frac{1}{q}\partial_t \|\nabla\theta\|_{L^q}^q \leq c\frac{(q-1)}{\kappa}\|u\|_{L^\infty}^2 \|\nabla\theta\|_{L^q}^q$$

or

$$\partial_t \|\nabla\theta\|_{L^q} \leq c(\frac{q-1}{\kappa})\|u\|_{L^\infty}^2 \|\nabla\theta\|_{L^q}.$$

Using the Lemma 1.5.1 again, we get

$$\|u\|_{L^\infty} \leq c\|u\|_{L^2} + c(1+\|w\|_{L^2})\sqrt{\log(1+\|\nabla w\|_{L^p})}$$

Therefore,

$$Z(t) = \|w\|_{L^q} + \|\nabla\theta\|_{L^q}$$

obeys

$$\frac{d}{dt}Z(t) \leq c(1+\|w\|_{L^2})Z(t)\log(1+Z(t)) + c$$

This allows us to obtain a global bound for $Z(t)$. This completes the proof of Theorem 1.5.2.

1.6 Horizontal dissipation

The results presented in this section are from the work of R. Danchin and M. Paicu [25] and from that of Larinos, Lunasin and Titi [36].

Consider the IVP for the 2D Boussinesq equations with horizontal dissipation

$$\begin{cases}
\partial_t u + (u\cdot\nabla)u = -\nabla p + \nu u_{xx} + \theta\overrightarrow{e_2}, \\
\nabla\cdot u = 0, \\
\partial_t\theta + (u\cdot\nabla)\theta = 0, \\
u(x,0) = u_0(x),\ \theta(x,0) = \theta_0(x)
\end{cases} \tag{1.19}$$

(1.19) has been shown to possess a unique global solution for suitable (u_0, θ_0) and the following theorem combines the results of R. Danchin and M. Paicu [25] and of Larinos, Lunasin and Titi [36].

Theorem 1.6.1. *Let $u_0 \in H^1(\mathbb{R}^2)$ and $\nabla \cdot u_0 = 0$. Assume $\omega_0 = \nabla \times u_0 \in \sqrt{L}$, namely*

$$\sup_{q\geq 2} \frac{\|\omega_0\|_{L^q}}{\sqrt{q}} < +\infty$$

Let $\theta_0 \in L^2 \cap L^\infty$. Then the IVP (1.19) has a unique solution (u, θ) satisfying

$$u \in L^\infty_{loc}([0,\infty); H^1), \quad \omega \in L^\infty_{loc}([0,\infty); \sqrt{L}), \quad u_2 \in L^2_{loc}([0,\infty); H^2),$$

$$\theta \in C_b([0,\infty); L^2), \quad \theta \in L^\infty([0,\infty); L^\infty).$$

In addition, if $\theta_0 \in H^s$ with $s \in (1/2, 1)$, then

$$\theta \in L^\infty_{loc}([0,\infty); H^{s-}). \tag{1.20}$$

for any $\epsilon > 0$.

Remark 1.6.1. *The paper of R. Danchin and M. Paicu [25] originally assumed that $\theta_0 \in H^s$ with $s \in (1/2, 1)$ to show the uniqueness. Later Larinos, Lunasin and Titi [36] was able to prove the uniqueness without this assumption.*

Remark 1.6.2. *The regularity result for θ in (1.20) comes from a typical losing type estimate.*

The proof of Theorem 1.6.1 is divided into several steps, which can be accomplished through the following subsections.

1.6.1 Global weak solution

This subsection proves the global existence of weak solutions in a very weak functional setting via Friedriches' Method. This method cuts off the high frequencies and thus smooths the functions. The global existence result can be stated as follows.

Theorem 1.6.2. *Let $\theta_0 \in L^2 \cap L^\infty$ and $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$. Then (1.19) has a global weak solution (u, θ) satisfying $\theta \in L^\infty([0,\infty); L^2 \cap L^\infty)$, $u \in L^\infty_{loc}([0,\infty); H^1)$, $u_2 \in L^2_{loc}([0,\infty); H^2)$.*

Proof. (Friedriches' Method) Let $n \in \mathbb{N}$ and define

$$L^2_n = \{f \in L^2(\mathbb{R}^2) | \operatorname{supp}\widehat{f} \subset B(0,n)\},$$

$$J_n f = (\chi_{B(0;n)}\widehat{f})^\vee,$$

where $\widehat{f}$ and $f^\vee$ denotes the Fourier and the inverse Fourier transforms, respectively, and $\chi_{B(0;n)}$ is the characteristic function on $B(0,n)$. Clearly, $J_n f \in H^\infty = \cap_{s\geq 0} H^s$. Consider the equations

$$\begin{cases} \partial_t \theta + J_n \nabla \cdot (J_n u J_n \theta) = 0, \\ \partial_t u + \mathbb{P} J_n \nabla \cdot (J_n \mathbb{P} u \otimes J_n \mathbb{P} u) = \nu J_n \mathbb{P} \partial_{xx} u + J_n \mathbb{P}(\theta \overrightarrow{e_2}), \\ u(x,0) = J_n u_0, \quad \theta(x,0) = J_n \theta_0, \end{cases} \tag{1.21}$$

where $\mathbb{P}$ denotes the Leray projection. By the Picard theorem, there exists $T^* > 0$ and a solution $(u, \theta) \in C^1([0,T), L^2_n)$ (of course the solution depends on n, but here we just write (u,θ) for notational convenience). Noticing that $J_n f = f$ if $f \in L^2_n$ and $\mathbb{P}F = F$ if $\nabla \cdot F = 0$, we have

$$\begin{cases} \partial_t \theta + J_n \nabla \cdot (u\theta) = 0, \\ \partial_t u + \mathbb{P} J_n \nabla \cdot (u \otimes u) = \nu \partial_{xx} u + \mathbb{P}(\theta \overrightarrow{e_2}). \end{cases}$$

By the energy method

$$\|\theta\|_{L^2} \le \|J_n \theta_0\|_2 \le \|\theta_0\|_{L^2},$$

$$\|u\|_{L^2}^2 + 2\nu \int_0^t \|\partial_x u\|_{L^2}^2 dt \le (\|u_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$

Taking the curl of the u-equation yields

$$\partial_t \omega + \mathbb{P} J_n (u \cdot \nabla \omega) = \nu \partial_{xx} \omega + \partial_x \theta$$

and thus

$$\|\omega\|_2^2 + 2\nu \int_0^t \|\partial_x \omega\|_2^2 d\tau \le \int_0^t \|\theta_0\|_2 \|\partial_x \omega\|_2 d\tau \le \nu \int_0^t \|\partial_x \omega\|_2^2 d\tau + \frac{C}{\nu} \int_0^t \|\theta_0\|_2^2 d\tau.$$

Therefore, $\theta \in L^\infty([0,\infty); L^2)$ and $u \in L^\infty([0,T]; H^1)$ for any $T > 0$. By the Picard Extension Theorem, (θ, u) is global in time and admits bounds that are uniform in n,

$$\theta^{(n)} \in L^\infty([0,\infty); L^2), \qquad u^{(n)} \in L^\infty([0,T]; H^1).$$

In addition, it can be shown that

$$\partial_t \theta^{(n)} \in L^\infty([0,T]; H^{-3=2}), \qquad \partial_t u^{(n)} \in L^\infty([0,T]; H^{-1})$$

Since $L^2 \hookrightarrow H^{-3=2}$ locally and $H^1 \hookrightarrow H^{-1}$ locally, the Aubin-Lions compactness lemma then implies $u^{(n)} \to u$ in H^L_{loc} for any $-1 \le L < 1$ and $\theta^{(n)} \to \theta$ in H^L_{loc} for any $-3/2 \le L < 0$. We can use these convergence to pass the limit in the weak formulation. This completes the proof.

1.6.2 Local classical solution

This subsection shows that (1.19) has a unique local classical solution.

Theorem 1.6.3. *Let $(u_0, \theta_0) \in H^s \times H^{s-1}$ with $s > 2$. Then there is $T > 0$ and a unique solution $(u,\theta) \in C([0,T), H^s \times H^{s-1})$ satisfying (1.19).*

Proof. We can either use the mollifier approach as in the book of Majda and Bertozzi [39] or Friedriches' approach above. The crucial part is a local priori bound. Define

$$J^s f = (1-\Delta)^{s=2} f \quad \text{or} \quad \widehat{J^s f}(\xi) = (1+|\xi|^2)^{\frac{s}{2}} \hat{f}(\xi).$$

Then clearly $\|J^s f\|_{L^2} = \|f\|_{H^s}$. It follows from the vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \partial_{x_1 x_1} \omega + \partial_{x_1} \theta$$

that

$$\frac{1}{2}\frac{d}{dt}\|J^{s-1}\omega\|_2^2 + \nu\|\partial_{x_1} J^{s-1}\omega\|_2^2 = K_1 + K_2,$$

where

$$\begin{aligned} K_1 &= \int \partial_{x_1} J^{s-1}\theta J^{s-1}\omega, \\ K_2 &= -\int (J^{s-1}(u\cdot\nabla\omega) - u\cdot\nabla J^{s-1}\omega) J^{s-1}\omega. \end{aligned}$$

K_1 and K_2 can be bounded as follows. By integration by parts,

$$|K_1| \le \frac{\nu}{2}\|\partial_{x_1} J^{s-1}\omega\|_2^2 + \frac{1}{2\nu}\|J^{s-1}\theta\|_2^2.$$

Applying the commutator estimates in Lemma 1.3.1 yields

$$\begin{aligned} |K_2| &\le \|J^{s-1}(u\cdot\nabla\omega) - u\cdot\nabla J^{s-1}w\|_2 \left\|J^{s-1}\omega\right\|_2 \\ &\le C(\|J^s u\|_2\|\omega\|_\infty + \|J^{s-1}\omega\|_2\|\nabla u\|_\infty)\|J^{s-1}\omega\|_2 \\ &\le C\,\|\nabla u\|_\infty \|J^{s-1}w\|_2^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|J^{s-1}\theta\|_2^2 &= -\int (J^{s-1}(u\cdot\nabla\theta) - u\cdot\nabla J^{s-1}\theta)J^{s-1}\theta \\ &\le C\|J^{s-1}\theta\|_2(\|J^s u\|_2\|\theta\|_\infty + \|J^{s-1}\theta\|_2\|\nabla u\|_\infty) \\ &\le C\|\theta\|_\infty(\|J^{s-1}\omega\|_2^2 + \|J^{s-1}\theta\|_2^2) + C\|\nabla u\|_\infty\|J^{s-1}\theta\|_2^2 \end{aligned}$$

Therefore, $Y(t) = \|J^{s-1}\omega\|_2^2 + \|J^{s-1}\theta\|_2^2$ satisfies

$$\frac{d}{dt}Y(t) \le C\left(1 + ||\theta||_\infty + \|\nabla u\|_\infty\right) Y(t).$$

Since $s > 2$,

$$||\nabla u||_\infty \le C\,||\nabla u||_{H^{s-1}} \le C\left|\left|J^{s-1}\omega\right|\right|_2.$$

and consequently

$$\frac{d}{dt}Y(t) \le C\left(1 + ||\theta||_\infty + \sqrt{Y(t)}\right) Y(t).$$

This inequality implies that $\exists T^*$ such that $Y(t) \le C$ for $t < T^*$. This completes the proof.

1.6.3 Global classical solution

This section establishes a global bound for $\|u\|_{H^s}$ and $\|\theta\|_{H^{s-1}}$, which allows us to extend the local solution in the previous subsection to a global one.

Theorem 1.6.4. *Assume that* $(u_0, \theta_0) \in H^s \times H^{s-1}$ *with* $s > 2$. *Then (1.19) has a unique global solution* $(u, \theta) \in C([0, \infty), H^s \times H^{s-1})$.

Proof. It suffices to obtain a global bound for (u, θ) in $H^s \times H^{s-1}$. From the previous proof, $Y(t) = \|J^{s-1}\omega\|_2^2 + \|J^{s-1}\theta\|_2^2$ satisfies

$$\frac{d}{dt}Y(t) \le C(1 + \|\theta\|_\infty + \|\nabla u\|_\infty)\, Y(t). \tag{1.22}$$

The trick is still to control $\|\nabla u\|_\infty$ through the following logarithmic interpolation inequality, for $\sigma > 1$,

$$\|f\|_\infty \le \sup_{q\ge 2} \frac{\|f\|_{L^q}}{\sqrt{q}} \sqrt{\log(1 + \|f\|_{H^\sigma})},$$

as proven in Lemma 1.6.1 below. In particular, for $s > 2$, we have

$$\begin{aligned} ||\nabla u||_\infty &\le C \sup_{q\ge 2} \frac{||\nabla u||_q}{\sqrt{q}} \sqrt{\log(1 + ||\nabla u||_{H^{s-1}})} \\ &\le C \sup_{q\ge 2} \frac{||\nabla u||_q}{\sqrt{q}} \sqrt{\log(1 + ||\omega||_{H^{s-1}})}. \end{aligned} \tag{1.23}$$

The goal is then to show that, for any $T > 0$,

$$\int_0^T \sup_{q\ge 2} \frac{||\nabla u||_q}{\sqrt{q}}\, dt < \infty. \tag{1.24}$$

This is accomplished through two steps. First, we have

$$\|\omega(T)\|_2^2 + 2\nu \int_0^T \|\partial_{x_1}\omega\|_2^2\, dt \le \frac{2T}{\nu}\|\theta_0\|_2^2$$

and, as in (1.14),

$$\sup_{q\ge 2} \frac{||\omega||_q}{\sqrt{q}} \le \sup_{q\ge 2} \frac{||\omega_0||_q}{\sqrt{q}} + \sqrt{\frac{2t}{\nu}}\|\theta_0\|_{L^2\cap L^\infty}.$$

We caution that the inequality $\|\nabla u\|_q \le \frac{c\,q^2}{q-1}\|\omega\|_q$ does not really help. Instead, we use the imbedding inequality, for a constant C independent of q,

$$\sup_{q\ge 2} \frac{||f||_q}{\sqrt{q}} \le C\,\|f\|_{H^1}$$

and the simple fact that $\|\partial_{x_1}\omega\|_2 = \|\nabla\partial_{x_1}u\|_2$, we have

$$\int_0^T \sup_{q\ge 2} \frac{\|\partial_{x_1}u\|_q}{\sqrt{q}}\, dt \le C \int_0^T \|\partial_{x_1}\nabla u\|_2 dt = C \int_0^T \|\partial_{x_1}\omega\|_2 dt < \infty.$$

Due to the divergence-free condition $\partial_{x_1} u_1 + \partial_{x_2} u_2 = 0$, we also have

$$\int_0^T \sup_{q\geq 2} \frac{\|\partial_{x_2} u_2\|_q}{\sqrt{q}}\, dt < \infty.$$

In addition, $\partial_{x_2} u_1 = \partial_{x_1} u_2 - \omega$ and thus

$$\int_0^T \sup_{q\geq 2} \frac{\|\partial_{x_2} u_1\|_q}{\sqrt{q}}\, dt \leq \int_0^T \sup_{q\geq 2} \frac{\|\partial_{x_1} u_2\|_q}{\sqrt{q}}\, dt + \int_0^T \sup_{q\geq 2} \frac{\|\omega\|_q}{\sqrt{q}}\, dt < \infty.$$

Thus we have proven (1.24). Combining (1.22), (1.23) and (1.24) yields the desired global bound. This completes the proof of Theorem 1.6.4. □

We now provide the lemma used in the proof of Theorem 1.6.4.

Lemma 1.6.1. *For $\sigma > 1$,*

$$\|f\|_\infty \leq C(\sigma) \sup_{q\geq 2} \frac{\|f\|_{L^q}}{\sqrt{q}} \sqrt{\log(1 + \|f\|_{H^\sigma})},$$

$$\sup_{q\geq 2} \frac{\|f\|_{L^q}}{\sqrt{q}} \leq C\, \|f\|_{H^1},$$

where C's are constants.

1.6.4 Proof of Theorem 1.6.1

This subsection outlines the proof of Theorem 1.6.1.

Proof of Theorem 1.6.1. It consists of three main steps. The first step proves the existence, the second proves the uniqueness while the last step provides the losing estimates. The first two steps are provided here, but the losing estimates are separated into another subsection.

To prove the existence, we first regularize the initial condition. For $\epsilon > 0$, set

$$J\ f = \rho\ * f$$

where $\rho\ = \frac{1}{2}\rho_0(\frac{x}{})$ with $\rho_0 \in C_0^\infty(\mathbb{R}^2)$, $\int_{\mathbb{R}^2} \rho_0(x)\, dx = 1$ and

$$\rho(x) = \begin{cases} 1, & \text{if} \quad |x| \leq \frac{1}{2}, \\ 0, & \text{if} \quad |x| > 1. \end{cases}$$

Consider (1.19) with the initial data $u\ (x,0) = J\ * u_0$ and $\theta\ (x,0) = J\ * \theta_0$. Since, for any $s > 0$,

$$||u_0||_{H^s} \leq C\frac{1}{\epsilon^{s-1}}\, ||u_0||_{H^1}\, , \quad ||\theta_0||_{H^s} \leq C\frac{1}{\epsilon^s}\, ||\theta_0||_{L^2}\, ,$$

where C is a constant independent of ϵ. By Theorem 1.6.4, there exists a unique solution

$$(u\ ,\theta\ ) \in C([0,\infty); H^s \times H^{s-1}).$$

Since $(u\ ,\theta\ )$ admits a global uniform bound in $H^1 \times L^2$, $(u\ ,\theta\ ) \rightharpoonup (u,\theta)$ in $H^1 \times L^2$ and (u,θ) is a weak solution according to Theorem 1.6.2.

We now show the uniqueness. R. Danchin and M. Paicu [25] assumed that

$$\theta_0 \in H^s \cap L^\infty \quad \text{with} \quad s \in \left(\frac{1}{2}, 1\right).$$

Larios, Lunasin and Titi [36] obtained the uniqueness by just assuming $\theta_0 \in L^2 \cap L^\infty$. What is presented here is a modified version of Larios, Lunasin and Titi and the essential idea is the Yudovich approach. Let $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$ be two solutions. Then the difference $(\widetilde{u}, \widetilde{\theta})$,

$$\widetilde{u} = u^{(1)} - u^{(2)}, \quad \widetilde{\theta} = \theta^{(1)} - \theta^{(2)},$$

satisfies

$$\begin{cases} \partial_t \widetilde{u} + u^{(1)} \cdot \nabla \widetilde{u} + \widetilde{u} \cdot \nabla u^{(2)} = \partial_{xx} \widetilde{u} - \nabla \widetilde{p} + \widetilde{\theta} \vec{e}_2, \\ \partial_t \widetilde{\theta} + u^{(1)} \cdot \nabla \widetilde{\theta} + \widetilde{u} \cdot \nabla \theta^{(2)} = 0. \end{cases}$$

The second equation demands too much regularity on θ. To exchange into a variable with less regularity requirement, we introduce

$$\Delta \xi^{(1)} = \theta^{(1)}, \quad \Delta \xi^{(2)} = \theta^{(2)}, \quad \widetilde{\xi} = \xi^{(1)} - \xi^{(2)}.$$

Clearly, $\widetilde{\xi}$ satisfies

$$\partial_t \Delta \widetilde{\xi} + u^{(1)} \cdot \nabla (\Delta \widetilde{\xi}) + \widetilde{u} \cdot \nabla \Delta \xi^{(2)} = 0.$$

The goal is to show $\widetilde{u} \equiv 0$ and $\widetilde{\xi} \equiv 0$. It follows from simple energy estimates that

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{u}\|_2^2 + \|\partial_x \widetilde{u}\|_2^2 = -\int \widetilde{u} \cdot \nabla u^{(2)} \cdot \widetilde{u} + \int \Delta \widetilde{\xi} \vec{e}_2 \cdot \widetilde{u},$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla \widetilde{\xi}\|_2^2 = -\int u^{(1)} \cdot \nabla \Delta \widetilde{\xi}\, \widetilde{\xi} - \int \widetilde{u} \cdot \nabla \Delta \xi^{(2)}\, \widetilde{\xi}.$$

We now estimates the terms on the right.

$$\begin{aligned} \int \Delta \widetilde{\xi}\, \vec{e_2} \cdot u &= \int \Delta \widetilde{\xi}\, \widetilde{u}_2 = \int (\partial_{x_1 x_1} + \partial_{x_2 x_2}) \widetilde{\xi}\, \widetilde{u}_2 \\ &= -\int \partial_{x_1} \widetilde{\xi} \partial_{x_1} \widetilde{u}_2 + \int \partial_{x_2} \widetilde{\xi} \partial_{x_1} \widetilde{u}_1 \\ &\le \frac{1}{2}\|\nabla \widetilde{\xi}\|_2^2 + \frac{1}{2}\|\partial_{x_1} \widetilde{u}\|_2^2. \end{aligned}$$

Integrating by parts and noticing that $\theta^{(2)} = \Delta\widetilde{\xi}^{(2)}$, we have

$$\begin{aligned} -\int \widetilde{u}\cdot\nabla\Delta\xi^{(2)}\widetilde{\xi} &= \int \widetilde{u}\cdot\nabla\widetilde{\xi}\Delta\widetilde{\xi}^{(2)} \\ &\le \frac{1}{2}\|\theta^{(2)}\|_\infty(\|\widetilde{u}\|_2^2 + \|\nabla\widetilde{\xi}\|_2^2). \end{aligned}$$

For notational convenience, we write

$$J_1 = -\int \widetilde{u}\cdot\nabla u^{(2)}\cdot\widetilde{u}, \qquad J_2 = -\int u^{(1)}\cdot\nabla\Delta\widetilde{\xi}\ \widetilde{\xi}.$$

Clearly

$$|J_1| \le \|\widetilde{u}\|_\infty^{\frac{2}{p}} \int |\nabla u^{(2)}||\widetilde{u}|^{2-\frac{2}{p}} \le \|\widetilde{u}\|_\infty^{\frac{2}{p}} \|\nabla u^{(2)}\|_p \|\widetilde{u}\|_2^{2-\frac{2}{p}}$$

We know that

$$\sup_{p\ge 2} \frac{\|\nabla u^{(2)}\|_p}{p} \le C_0$$

for C_0 independent of p. Also we have $\|\widetilde{u}\|_\infty \le M \equiv C\,\|\widetilde{u}\|_2^a\,\|\nabla\widetilde{u}\|_q^{1-a}$ for any $q > 2$ and $a = \frac{q-2}{2(q-1)}$. Therefore,

$$|J_1| \le C_0\, p\, M^{\frac{2}{p}}\, \|\widetilde{u}\|_2^{2-\frac{2}{p}}.$$

Using the simple fact that a function $f(x) = xA^{\frac{2}{x}}$ has the minimum $2e\log A$, we have

$$|J_1| \le 2C_0\,(\log M - \log\|\widetilde{u}\|_2)\|\widetilde{u}\|_2^2.$$

Integrating by parts yields

$$\begin{aligned} J_2 &= \int u^{(1)}\cdot\nabla\widetilde{\xi}\,\Delta\widetilde{\xi} \\ &= \int u^{(1)}\cdot\nabla\widetilde{\xi}\,\partial_k\partial_k\widetilde{\xi} \\ &= -\int \partial_k u^{(1)}\cdot\nabla\widetilde{\xi}\,\partial_k\widetilde{\xi} - \int u^{(1)}\cdot\nabla\partial_k\widetilde{\xi}\,\partial_k\widetilde{\xi} \\ &= -\int \partial_k u^{(1)}\cdot\nabla\widetilde{\xi}\,\partial_k\widetilde{\xi}. \end{aligned}$$

Therefore,

$$\begin{aligned} |J_2| &\le \int |\nabla u^{(1)}|\,|\nabla\widetilde{\xi}|^2 \\ &\le \|\nabla\widetilde{\xi}\|_\infty^{\frac{2}{p}} \int |\nabla u^{(1)}||\nabla\widetilde{\xi}|^{2-\frac{2}{p}}. \end{aligned}$$

Since $\|\nabla\widetilde{\xi}\|_\infty \le M$,

$$\begin{aligned} |J_2| &\le CpM^{\frac{2}{p}}\|\nabla\widetilde{\xi}\|_2^{2-\frac{2}{p}} \\ &\le C(\log M - \log\|\nabla\widetilde{\xi}\|_2)\,\|\nabla\widetilde{\xi}\|_2^2. \end{aligned}$$

Combining the estimates allows us to conclude that $X(t) = \|\widetilde{u}\|_2^2 + \|\nabla\widetilde{\xi}\|_2^2$ satisfies

$$\frac{d}{dt}X + \|\partial_x\widetilde{u}\|_2^2 \le CX + C\,(\log M - \log X)\,X,$$

where C's are constants.

$$\begin{aligned} \frac{d}{dt}X - C\,X &\le C\,(\log M - \log X)X \\ \frac{d}{dt}(e^{-C\,t}X) &\le Ce^{-C\,t}(\log M - \log X)X \\ X(t) &\le e^{C\,t}X(0) + C\int_0^t e^{C\,(t-\ )}(\log M - \log X)X\,d\tau. \end{aligned}$$

Since $X(0) = 0$, we have $X(t) \equiv 0$ for any $t > 0$ by Osgood inequality (see Lemma 1.6.2 below). This completes the proof of Theorem 1.6.1. □

Lemma 1.6.2 (Osgood Inequality)**.** *Let $\rho_0 \ge 0$ be a constant and $\alpha(t) \ge 0$ be a continuous function. Assume*

$$\rho(t) \le \rho_0 + \int_0^t \alpha(\tau)w(\rho(\tau))d\tau,$$

where w satisfies

$$\int_1^\infty \frac{1}{w(r)}dr = \infty.$$

Then $\rho_0 = 0$ implies $\rho \equiv 0$, and $\rho_0 > 0$ implies that

$$-\Omega(\rho(t)) + \Omega(\rho_0) \le \int_0^t \alpha(\tau)d\tau,$$

where $\Omega(\rho) = \int^1 \frac{dr}{w(r)}$.

In particular, the Osgood inequality applies to $w(\rho) = \rho\log\frac{M}{}$ since

$$\int_1^\infty \frac{1}{r(\log M - \log r)}dr = \infty.$$

1.6.5 Losing regularity estimates for θ

We stated in Theorem 1.6.1 that, if $\theta_0 \in H^s \cap L^\infty$ with $s \in \left(\frac{1}{2}, 1\right)$, then $\theta \in H^{s-\ }$ for any $\epsilon > 0$. The solution is no longer in the same functional setting as the initial data and the loss of regularity is due to the velocity is not Lipschitz. Other losing type estimates can be found in the book of H. Bahouri, J.-Y. Chemin and R. Danchin [4].

Theorem 1.6.5. *Consider*

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = f & x \in \mathbb{R}^2, \qquad t > 0, \\ \rho(0) = \rho_0, & x \in \mathbb{R}^2 \end{cases} \tag{1.25}$$

Assume $\rho_0 \in B^s_{2;\infty}(\mathbb{R}^2)$ *with* $-1 < s < 1$, $f \in L^1([0,T]; B^s_{2;\infty})$, $\nabla \cdot u = 0$ *and*

$$\sup_{N \geq 2} \frac{\|\nabla S_N u\|_{L^\infty}}{\sqrt{N}} \leq V(t)$$

for some $V \in L^1(0,T)$. *Then for any* $\epsilon \in (0, \frac{1+s}{2})$

$$\|\rho\|_{B^{s-\epsilon}_{2,\infty}} \leq C e^{\frac{C_1}{\epsilon}\left(\int_0^t V(\ )d\ \right)^2} \left(\|\rho_0\|_{B^s_{2,\infty}} + \|f\|_{L^1_t(B^s_{2,\infty})}\right),$$

where C *and* C_1 *are constants independent of* ϵ.

Proof. Applying Δ_j to (1.25) yields

$$\partial_t \Delta_j \rho = \Delta_j f - \Delta_j (u \cdot \nabla \rho).$$

Using the notion of paraproducts, we write

$$\begin{aligned} \Delta_j (u \cdot \nabla \rho) &= J_1 + J_2 + J_3 + J_4 + J_5 \\ &\equiv \sum_{|k-j| \leq 2} [\Delta_j (S_{k-1} u \cdot \nabla \Delta_k \rho) - S_{k-1} u \cdot \nabla \Delta_j \Delta_k \rho] \\ &+ \sum_{|k-j| \leq 2} [(S_{k-1} u - S_j u) \nabla \Delta_j \Delta_k \rho] + S_j u \nabla \Delta_j \rho \\ &+ \sum_{|k-1| \leq 2} \Delta_j (\Delta_k u \nabla S_{k-1} \rho) + \sum_{k \geq j-1} \Delta_j (\Delta_k u \cdot \nabla \tilde{\Delta}_k \rho) \end{aligned}$$

where $\tilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. Taking the inner product with $\Delta_j \rho$ and integrating over the space leads to

$$\frac{d}{dt} \|\Delta_j \rho\|_2^2 \leq \|\Delta_j f\|_2 \|\Delta_j \rho\|_2 + (\|J_1\|_2 + \|J_2\|_2 + \|J_4\|_2 + \|J_5\|_2) \|\Delta_j \rho\|_2,$$

where we have used the fact that

$$\int J_3 \, \Delta_j \rho \, dx = \int (S_j u \cdot \nabla \Delta_j \rho) \Delta_j \rho \, dx = 0.$$

The norms $\|J_1\|_2$ through $\|J_5\|_2$ can be bounded as follows.

$$\|J_1\|_2 \le C \sum_{|k-j|\le 2} \|\nabla S_{k-1}u\|_{L^\infty} \|\Delta_k \rho\|_2, \quad \|J_2\|_2 \le C \sum_{|k-j|\le 2} \|\Delta_k u\|_{L^\infty} 2^j \|\Delta_j \rho\|_2,$$

$$\|J_4\|_2 \le C \sum_{|k-j|\le 2} \|\Delta_k u\|_{L^\infty} \|\nabla S_{k-1}\rho\|_2, \quad \|J_5\|_2 \le \sum_{k\ge j-1} 2^j \|\Delta_k u\|_{L^\infty} \|\tilde{\Delta}_k \rho\|_2.$$

For fixed j, the summation over k satisfying $|k-j| \le 2$ contains only a finite number of terms and can be bounded by a constant multiple of the term with $j = k$. Thus we have obtained

$$\frac{d}{dt}\|\Delta_j \rho\|_2 \le \|\Delta_j f\|_2 + C\|\nabla S_{j-1}u\|_{L^\infty} \|\Delta_j \rho\|_2 + 2^j \|\Delta_j u\|_{L^\infty} \|\Delta_j \rho\|_2$$

$$+C\|\Delta_j u\|_{L^\infty} \|\nabla S_{j-1}\rho\|_2 + C \sum_{k\ge j-1} 2^j \|\Delta_k u\|_{L^\infty} \|\Delta_k \rho\|_2.$$

Furthermore, $\|\nabla S_{j-1}u\|_{L^\infty} \le \sqrt{j+2}\, V(t)$ and, by the lower bound part of Bernstein's inequality, $2^j \|\Delta_j u\|_{L^\infty} \le \sqrt{j+2}\, V(t)$ for $j \ge 0$. Also, for any $-1 < \sigma < 1$,

$$\begin{aligned} \|\nabla S_{j-1}\rho\|_2 &= \sum_{m\le j-2} 2^m \|\Delta_m \rho\|_2 \\ &\le C 2^{j(1-\ )} \sum_{m\le j-2} 2^{(m-j)(1-\ )} 2^{\ m} \|\Delta_m \rho\|_2 \\ &= C 2^{j(1-\ )} \|\rho\|_{B^\sigma_{2,\infty}} \end{aligned}$$

and

$$\begin{aligned} &\sum_{k\ge j-1} 2^j \|\Delta_k u\|_{L^\infty} \|\Delta_k \rho\|_2 \\ =\ & 2^{j(1-\ )} \sum_{k\ge j-1} 2^{\ (j-k)} \|\nabla \Delta_k u\|_{L^\infty} \sup_{k\ge j-1} 2^{k\ } \|\Delta_k \rho\|_2 \\ \le\ & C 2^{-\ j} \|\rho\|_{B^\sigma_{2,\infty}} \sum_{k\ge j-1} 2^{(1+\ )(j-k)} \sqrt{k+2} \frac{\|\nabla \Delta_k u\|_{L^\infty}}{\sqrt{k+2}} \\ \le\ & C 2^{-\ j} \|\rho\|_{B^\sigma_{2,\infty}} \sup_{k\ge j-1} \frac{\|\nabla \Delta_k u\|_{L^\infty}}{\sqrt{k+2}} \sqrt{j+2} \sum_{k\ge j-1} \frac{\sqrt{k+2}}{\sqrt{j+2}} 2^{(1+\ )(j-k)} \\ \le\ & C 2^{-\ j} \|\rho\|_{B^\sigma_{2,\infty}} V(t) \sqrt{j+2}. \end{aligned}$$

Putting the estimates together, we find

$$\frac{d}{dt}\|\Delta_j \rho\|_2 \le \|\Delta_j f\|_2 + C\, 2^{-\ j}\, V(t) \sqrt{j+2}\, \|\rho\|_{B^\sigma_{2,\infty}}.$$

Now we take

$$\sigma_t = s - \epsilon \frac{\int_0^t V(s)ds}{\int_0^T V(s)ds} \quad \text{or} \quad \sigma_t = s - \eta \int_0^t V(s)ds \quad \text{with} \quad \eta = \frac{\epsilon}{\int_0^t V(s)ds}.$$

Clearly $\sigma_t \in (-1,1)$. We substitute σ_t for σ and integrate in time to obtain

$$\|\Delta_j \rho\|_2 \le \|\Delta_j \rho_0\|_2 + \int_0^t \|\Delta_j f\|_2 d\tau + C\sqrt{j+2}\int_0^t 2^{-\ \ \tau j} V(\tau)\|\rho\|_{B^{\sigma_\tau}_{2,\infty}} d\tau.$$

Multiplying by $2^{\ \ tj}$ and noticing that $\sigma_t - \sigma\ \ = -\eta \int^t V(s)ds$, we have

$$2^{\ \ tj}\|\Delta_j \rho\|_2 \le \|\rho_0\|_{B^{\sigma_t}_{2,\infty}} + \int_0^t \|f\|_{B^{\sigma_t}_{2,\infty}} d\tau + C\int_0^t \sqrt{j+2}V(\tau)\, 2^{-\ \ j\int_\tau^t V(s)ds}\|\rho\|_{B^{\sigma_\tau}_{2,\infty}} d\tau.$$

Choose the smallest integer $j_0 > 0$ such that

$$\frac{4}{(\ln(2))^2\eta^2} \le j_0 + 2. \tag{1.26}$$

Then, for $j \ge j_0$,

$$C\int_0^t \sqrt{j+2}\, V(\tau)e^{-\ \ j\int_\tau^t V(s)ds} d\tau < \frac{1}{2}.$$

In fact, by direct integration,

$$\begin{aligned} A &\equiv C\int_0^t \sqrt{j+2}V(\tau)2^{-j\ \ \int_\tau^t V(s)ds} d\tau \\ &= \frac{C\sqrt{j+2}}{j\eta}\int_0^t \eta j V(\tau) 2^{-j\ \ \int_\tau^t V(s)ds} d\tau \\ &= \frac{C\sqrt{j+2}}{j\eta\ln(2)}\int_0^t d(2^{-j\ \ \int_\tau^t V(s)ds}) \\ &= \frac{C\sqrt{j+2}}{j\eta\ln(2)}(1 - 2^{-j\ \ \int_0^t V(s)ds}) < \frac{1}{2}. \end{aligned}$$

Fix $t_0 \le T$ and assume $t \le t_0$. We now take the supremum over $j \ge -1$

$$\begin{aligned} \sup_{j\ge -1} 2^{j\ \ t}\|\triangle_j \rho\|_{L^2} &\le \|\rho_0\|_{B^s_{2,\infty}} + \int_0^t \|f(\tau)\|_{B^s_{2,\infty}} d\tau \\ &\quad + C\left(\sup_{j\ge j_0} + \sup_{j\le j_0}\right)\int_0^t \sqrt{j+2}V(\tau)2^{-\ \ j\int_\tau^t V(s)ds}\|\rho\|_{B^{\sigma_\tau}_{2,\infty}} d\tau \\ &\le \|\rho_0\|_{B^s_{2,\infty}} + \int_0^t \|f(\tau)\|_{B^s_{2,\infty}} d\tau + \frac{1}{2}\sup_{\ \in[0;t_0]}\|\rho(\tau)\|_{B^{\sigma_\tau}_{2,\infty}} \\ &\quad + c\sqrt{j_0+2}\int_0^t V(\tau)\|\rho(\tau)\|_{B^{\sigma_\tau}_{2,\infty}} d\tau. \end{aligned}$$

Since this holds for any $t \le t_0 \le T$, we take the supremum over $t \in [0, t_0]$ to obtain

$$\begin{aligned} \sup_{t\in[0;t_0]} \|\rho(t)\|_{B^{\sigma_t}_{2,\infty}} &\le 2\|\rho_0\|_{B^s_{2,\infty}} + 2\int_0^{t_0} \|f(\tau)\|_{B^s_{2,\infty}} d\tau \\ &\quad + C\sqrt{j_0+2}\int_0^{t_0} V(\tau)\|\rho(\tau)\|_{B^{\sigma_\tau}_{2,\infty}} d\tau. \end{aligned}$$

Then

$$z(t) = \sup_{\in[0;t]} \|\rho(\tau)\|_{B^{\sigma_\tau}_{2,\infty}}$$

obeys

$$z(t_0) \le 2\|\rho_0\|_{B^s_{2,\infty}} + 2\int_0^{t_0} \|f(\tau)\|_{B^s_{2,\infty}} d\tau + C\sqrt{j_0+2}\int_0^{t_0} V(\tau) z(\tau) d\tau.$$

Applying Gronwall's inequality and recalling that

$$\sqrt{j_0+2} \le \frac{4}{\ln 2}\frac{1}{\epsilon}\int_0^T V(\tau)\, d\tau,$$

we obtain, for any $t \le T$,

$$z(t) \le 2\left[\|\rho_0\|_{B^s_{2,\infty}} + \int_0^t \|f(\tau)\|_{B^s_{2,\infty}} d\tau\right] \exp\left(\frac{C}{\epsilon}\left(\int_0^T V(\tau) d\tau\right)^2\right),$$

where C is independent of ϵ. This completes the proof of Theorem 1.6.5. □

1.7 Horizontal thermal diffusion

We work on the 2D Boussinesq equations with only horizontal diffusion

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = -\nabla p + \theta \overrightarrow{e_2}, \\ \partial_t \theta + (u\cdot\nabla)\theta = \partial_{x_1x_1}\theta, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.27}$$

where we have set the coefficient of $\partial_{x_1x_1}\theta$ to be 1, without loss of generality. (1.27) still possesses a unique global solution when (u_0, θ_0) is in a suitable functional setting. The following result is of R. Danchin and M. Paicu [25].

Theorem 1.7.1. *Assume $u_0 \in H^1, w_0 = \nabla\times u_0 \in L^\infty, \theta_0 \in H^1\cap L^\infty, |\partial_{x_1}|^{1+s}\theta_0 \in L^2$ for $s \in (0, \frac{1}{2}]$. Then (1.27) has a unique global solution (u,θ) satisfying*

$$u \in C([0,\infty); H^1), \quad w \in L^\infty_{loc}([0,\infty); L^\infty),$$
$$\theta \in C([0,\infty); H^1\cap L^\infty), \quad |\partial_{x_1}|^{1+s}\theta \in L^\infty_{loc}([0,\infty); L^2), \quad |\partial_{x_1}|^{2+s}\theta \in L^2_{loc}([0,\infty); L^2).$$

Here $|\partial_{x_1}|$ with $\beta \in \mathbb{R}$ is defined in terms of its Fourier transform,

$$|\partial_{x_1}|\ f(x) = \int e^{ix\cdot}\ |\xi_1|\ \hat{f}(\xi_1,\xi_2) d\xi_1 d\xi_2.$$

Proof. We will not provide the details, but rather briefly outline the major steps. The equations give us for free the global bounds on $\|\theta\|_{L^2\cap L^\infty}$ and $\|u\|_{H^1}$. The first real step is to obtain a global bound for $\|\nabla\theta\|_2$. This can be established by writing the nonlinear term explicitly in terms of the partial derivatives in different directions and fully take advantage of the dissipation in the x-direction. The resulting estimate is given by

$$\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 + \|\partial_{x_1}\nabla\theta\|_{L^2}^2 \le B(t)\|\nabla\theta\|_{L^2}^2,$$

where $B(t)$ is integrable on $[0,\infty)$. Some special consequences of this inequality are

$$\|\omega(\cdot,T)\|_{L^4} \le C(T), \quad \|u(\cdot,T)\|_{L^\infty} \le C(T),$$

where $T>0$ is an arbitrarily fixed and $C(T)$ depends on the initial data and T. In fact, by the Sobolev embedding inequality

$$\|\partial_{x_1}\theta\|_{L^4} \le C\|\partial_{x_1}\theta\|_{L^2}^{1=2}\|\nabla\partial_{x_1}\theta\|_{L^2}^{1=2},$$

we have

$$\int_0^T \|\partial_{x_1}\theta\|_{L^4}^4\,dt \le C\sup_{t\in[0;T]}\|\partial_{x_1}\theta\|_{L^2}^2\int_0^T\|\partial_{x_1}\nabla\theta\|_{L^2}^2\,dt < \infty.$$

It then follows from the vorticity equation that

$$\frac{1}{4}\frac{d}{dt}\|\omega\|_{L^4}^4 = \int \partial_{x_1}\theta\omega|\omega|^2dx \le \|\partial_{x_1}\theta\|_{L^4}\|\omega\|_{L^4}^3.$$

Therefore,

$$\|\omega\|_{L^4} \le \|\omega_0\|_{L^4} + \int_0^t\|\partial_{x_1}\theta\|_{L^4}d\tau \le \|\omega_0\|_{L^4} + \left(\int_0^t\|\partial_{x_1}\theta\|_{L^4}^4d\tau\right)^{1=4} t^{3=4}.$$

Thus,

$$\|u\|_{L^\infty} \le \|u\|_{L^2}^{1=3}\|\nabla u\|_{L^4}^{2=3} \le C\,\|u\|_{L^2}^{1=3}\|\omega\|_{L^4}^{2=3} < \infty.$$

Next we show that, for $s\in(0,1/2]$,

$$|\partial_{x_1}|^{1+s}\theta \in L^\infty_{loc}([0,\infty);L^2), \quad |\partial_{x_1}|^{2+s}\theta \in L^2_{loc}([0,\infty);L^2). \tag{1.28}$$

Clearly, $|\partial_{x_1}|^{1+s}\theta$ satisfies

$$\partial_t|\partial_{x_1}|^{1+s}\theta - \partial_{x_1x_1}|\partial_{x_1}|^{1+s}\theta = -|\partial_{x_1}|^{1+s}(u\cdot\nabla\theta).$$

Taking the inner product with $|\partial_{x_1}^{1+s}|\theta$, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\||\partial_{x_1}|^{1+s}\theta\|_{L^2}^2 + \||\partial_{x_1}|^{2+s}\theta\|_{L^2}^2 &= -\int|\partial_{x_1}|^{1+s}(u\cdot\nabla\theta)\,|\partial_{x_1}|^{1+s}\theta\,dx\\ &= -\int|\partial_{x_1}|(u\cdot\nabla\theta)\,|\partial_{x_1}|^{1+2s}\theta\,dx\\ &= \int\frac{\partial_{x_1}}{|\partial_{x_1}|}(\partial_{x_1}u\cdot\nabla\theta + u\cdot\nabla\partial_{x_1}\theta)|\partial_{x_1}|^{1+2s}\theta dx.\end{aligned}$$

Since $s \in (0, 1/2]$,

$$\begin{aligned}\int \frac{\partial_{x_1}}{|\partial_{x_1}|}(u\cdot\nabla\partial_{x_1}\theta)|\partial_{x_1}|^{1+2s}\theta dx &\leq \|u\|_{L^\infty}\|\nabla\partial_{x_1}\theta\|_{L^2}\||\partial_{x_1}|^{1+2s}\theta\|_{L^2}\\ &\leq \|u\|_{L^\infty}\|\partial_{x_1}\nabla\theta\|_{L^2}^2.\end{aligned}$$

Writing $\partial_{x_1}u\cdot\nabla\theta = \partial_{x_1}u_1\partial_{x_1}\theta + \partial_{x_1}u_2\partial_{x_2}\theta$, we have

$$\begin{aligned}\int \frac{\partial_{x_1}}{|\partial_{x_1}|}(\partial_{x_1}u\cdot\nabla\theta)\,|\partial_{x_1}|^{1+2s}\theta dx &\leq \|\omega\|_{L^4}\|\partial_{x_1}\theta\|_{L^4}\||\partial_{x_1}|^{1+2s}\theta\|_{L^2}\\ &\quad + \int \frac{\partial_{x_1}}{|\partial_{x_1}|}\partial_{x_1}u_2\partial_{x_2}\theta|\partial_{x_1}|^{1+2s}\theta.\end{aligned}$$

In addition,

$$\begin{aligned}\int \frac{\partial_{x_1}}{|\partial_{x_1}|}\partial_{x_1}u_2\partial_{x_2}\theta|\partial_{x_1}|^{1+2s}\theta &= -\int \frac{\partial_{x_1}}{|\partial_{x_1}|}u_2\partial_{x_1}\partial_{x_2}\theta|\partial_{x_1}|^{1+2s}\theta dx\\ &\quad -\int \frac{\partial_{x_1}}{|\partial_{x_1}|}|\partial_{x_1}|^s(u_2\partial_{x_2}\theta)|\partial_{x_1}|^{2+s}\theta dx\\ &\leq \|u_2\|_{L^\infty}\|\partial_{x_1}\nabla\theta\|_{L^2}\,\||\partial_{x_1}|^{1+2s}\theta\|_{L^2}\\ &\quad +\||\partial_{x_1}|^s(u_2\partial_{x_2}\theta)\|_{L^2}\||\partial_{x_1}|^{2+s}\theta\|_{L^2}\\ &\leq \|u_2\|_{L^\infty}^2\|\partial_{x_1}\nabla\theta\|_{L^2}^2 + \frac{1}{4}\||\partial_{x_1}|^{2+s}\theta\|_{L^2}^2\\ &\quad +C\|u\|_{H^1}^2(\|\partial_2\theta\|_{L^2}+\|\partial_1\partial_2\theta\|_{L^2})^2.\end{aligned}$$

Combining the estimates yield, for some $g \in L^1_{loc}([0,\infty))$,

$$\frac{d}{dt}\||\partial_{x_1}|^{1+s}\theta\|_{L^2}^2 + \||\partial_{x_1}|^{2+s}\theta\|_{L^2}^2 \leq g(t).$$

We thus have obtained (1.28). A special consequence is that

$$\omega \in L^\infty_{loc}([0,\infty); L^\infty).$$

This can be obtained by combining

$$\|\omega\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + \int_0^t \|\partial_{x_1}\theta\|_{L^\infty} d\tau$$

with the simple estimate of the following lemma.

Lemma 1.7.1. *If* $\frac{1}{s_1}+\frac{1}{s_2} < 2, \quad s_1 > 0, \quad s_2 > 0,$ *then*

$$\|f\|_{L^\infty} \leq C(\|f\|_{L^2} + \||\partial_{x_1}|^{s_1}f\|_{L^2} + \||\partial_{x_2}|^{s_2}f\|_{L^2}).$$

Applying this lemma with $s_1 = 1 + s, \quad s_2 = 1$, we have

$$\|\omega\|_{L^\infty} \le \|\omega_0\|_{L^\infty} + C\int_0^t (\|\partial_{x_1}\theta\|_{L^2} + \||\partial_{x_1}|^{2+s}\theta\|_{L^2} + \|\partial_{x_1}\partial_{x_2}\theta\|_{L^2})d\tau < \infty.$$

Trivially, interpolating between L^2 and L^∞ yields $\omega \in L^q$ for $q \in [2,\infty)$. More importantly,

$$\nabla u \in L^\infty_{loc}([0,\infty); L) \quad \text{or} \quad \sup_{t\in[0;T]} \sup_{q\ge 2} \frac{\|\nabla u\|_{L^q}}{q} \le C(T).$$

This completes the part for the existence and regularity part. Finally we show the uniqueness. This is a consequence of the Yudovich type argument. Let $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$ be two solutions. Then $\tilde{u} = u^{(1)} - u^{(2)}$ and $\tilde{\theta} = \theta^{(1)} - \theta^{(2)}$ satisfy

$$\begin{cases} \partial_t \tilde{u} + u^{(1)}\cdot\nabla\tilde{u} + \tilde{u}\cdot\nabla u^{(2)} = \theta\overrightarrow{e_2} \\ \partial_t\tilde{\theta} + u^{(1)}\cdot\nabla\tilde{\theta} + \tilde{u}\cdot\nabla\theta^{(2)} = \partial_{x_1x_1}\tilde{\theta} \end{cases}$$

The most difficult term we would encounter in the further estimates is

$$\int \tilde{u_2}\,\partial_2\theta^{(2)}\,\tilde{\theta}dx$$

One way to handle it is to use the identity

$$I = (I - \partial_{x_1}\partial_{x_2})^{-1} - (I - \partial_{x_1}\partial_{x_2})^{-1}\partial_{x_1}\partial_{x_2}.$$

We shall omit further details. This completes the proof of Theorem 1.7.1.

1.8 Vertical dissipation and vertical thermal diffusion

This section focuses on the global regularity problem for the 2D Boussinesq equations with vertical dissipation and vertical thermal diffusion. More precisely, we study the global existence and uniqueness of solutions to the initial-value problem of

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + \nu\partial_{yy}u + \theta\overrightarrow{e_2} \\ \partial_t\theta + u\cdot\nabla\theta = \kappa\partial_{yy}\theta \\ \nabla\cdot u = 0 \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.29}$$

We are able to establish the global existence and uniqueness in a recent work in collaboration with Cao [8]. We should also mention two previous papers [2] and [3] in which partial results have been established. The main results can be stated as follows.

Theorem 1.8.1. *Consider the IVP for the anisotropic Boussinesq equations with vertical dissipation (1.29). Let $\nu > 0$ and $\kappa > 0$. Let $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)$. Then, for any $T > 0$, (1.29) has a unique classical solution (u, v, θ) on $[0, T]$ satisfying*

$$(u, v, \theta) \in C([0, T]; H^2(\mathbb{R}^2)).$$

The rest of this sections proves this theorem. As we mentioned before, the local existence and uniqueness is not very hard to obtain. Therefore, our effort will be devoted to proving the global *a priori* H^2 bounds for the solution. For the sake of clarity, we should accomplish this big task in several subsections.

1.8.1 What is the difficulty?

We briefly explain the difficulty and point out a simple fact. The L^2-estimates are easy to get.

$$\|\theta(t)\|_2^2 + 2\kappa \int_0^T \|\partial_y \theta\|_2^2 \, dt \le \|\theta_0\|_2^2,$$

$$\|u(t)\|_2^2 + 2\nu \int_0^T \|\partial_y u\|_2^2 \, dt \le (\|u_0\|_2 + t \, \|\theta_0\|_2)^2.$$

But the global H^1-bound is hard to prove. Consider the equation for $\omega = \vec{\nabla} \times \vec{u}$, which satisfies

$$\partial_t \omega + \vec{u} \cdot \nabla \omega = \nu \partial_{yy} \omega + \partial_x \theta. \tag{1.30}$$

As a simple consequence,

$$\frac{1}{2} \frac{d}{dt} \int \omega^2 + \nu \int (\partial_y \omega)^2 = \int \partial_x \theta \omega.$$

In contrast to the horizontal dissipation case, the dissipation in the y-direction does not allow us to hide $\partial_x \theta$. Therefore, if we want to obtain a global bound for $\|\omega\|_2$, then we need to combine it with the estimate of $\nabla \theta$.

$$\frac{1}{2} \frac{d}{dt} \int |\nabla \theta|^2 + \kappa \int |\partial_y \nabla \theta|^2 = - \int \nabla \theta \cdot \nabla u \cdot \nabla \theta. \tag{1.31}$$

To make use of the dissipation in the y-direction, we write

$$\nabla \theta \cdot \nabla u \cdot \nabla \theta = \partial_x u (\partial_x \theta)^2 + \partial_x v \partial_x \theta \partial_y \theta + \partial_y u \partial_x \theta \partial_y \theta + \partial_y v (\partial_y \theta)^2. \tag{1.32}$$

To bound the terms on the right, we need a lemma.

Lemma 1.8.1. *Assume that $f, g, g_y, h, h_x \in L^2(\mathbb{R}^2)$. Then*

$$\iint |f \, g \, h| \, dxdy \le C \, \|f\|_2 \, \|g\|_2^{\frac{1}{2}} \, \|g_y\|_2^{\frac{1}{2}} \, \|h\|_2^{\frac{1}{2}} \, \|h_x\|_2^{\frac{1}{2}}. \tag{1.33}$$

This lemma allows us to bound some of the terms suitably. For example,

$$\begin{aligned}\left|\int \partial_y u \partial_x \theta \partial_y \theta\right| &\le C\,\|\partial_y u\|_2\,\|\partial_x \theta\|_2^{\frac{1}{2}}\,\|\|\partial_{xy}\theta\|_2^{\frac{1}{2}}\,\|\partial_y \theta\|_2^{\frac{1}{2}}\,\|\partial_{xy}\theta\|_2^{\frac{1}{2}}\\ &\le \frac{\kappa}{4}\|\partial_{xy}\theta\|_2^2 + C(\kappa)\,\|\partial_y u\|_2^2\,\|\partial_x\theta\|_2\,\|\partial_y\theta\|_2.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}\int \partial_x v \partial_x \theta \partial_y \theta &= -\int \theta\,(\partial_x v \partial_{xy}\theta + \partial_{xy} v \partial_x \theta)\\ &\le \|\theta_0\|_\infty\,\|\partial_x v\|_2\,\|\partial_{xy}\theta\|_2 + \|\theta_0\|_\infty\,\|\partial_x \theta\|_2\,\|\partial_{xy} v\|_2\end{aligned}$$

However, the term $\int \partial_x u\,(\partial_x\theta)^2$ can NOT be bounded suitably. But if we know

$$\int_0^T \|v(t)\|_{L^\infty}^2\,dt < \infty, \tag{1.34}$$

then we have, after integration by parts,

$$\begin{aligned}\int \partial_x u\,(\partial_x\theta)^2 &= -\int \partial_y v\,(\partial_x\theta)^2 = \int v \partial_x \theta\,\partial_{xy}\theta\\ &\le \frac{\kappa}{4}\|\partial_{xy}\theta\|_2^2 + C(\kappa)\,\|v(t)\|_{L^\infty}^2\,\|\partial_x\theta\|_2^2.\end{aligned}$$

Inserting the estimates above in (1.30) and (1.31), we are able to conclude that, if (1.34) holds, then

$$\|\omega\|_2^2 + \|\nabla\theta\|_2^2 + \nu\int(\partial_y\omega)^2 + \kappa\int|\partial_y\nabla\theta|^2 \le C(T).$$

Unfortunately it appears to be extremely hard to prove (1.34). Therefore, we have to solve this problem through a different route.

1.8.2 Key ingredients in the proof

This section presents the key ingredients in the proof as well as the proof of Theorem 1.8.1.

Proposition 1.8.1. *Assume $(u_0, v_0, \theta_0) \in H^2$. Let (u, v, θ) be the corresponding classical solution of (1.29). Then the quantity*

$$Y(t) = \|\omega\|_{H^1}^2 + \|\theta\|_{H^2}^2 + \|\omega^2 + |\nabla\theta|^2\|_2^2$$

satisfies

$$\begin{aligned}&\frac{d}{dt}Y(t) + \|\omega_y\|_{H^1}^2 + \|\theta_y\|_{H^2}^2 + \int(\omega^2 + |\nabla\theta|^2)\,\left(\omega^2 + |\nabla\theta_y|^2\right)\\ &\quad\le C\left(1 + \|\theta_0\|_\infty^2 + \|v\|_\infty^2 + \|u_y\|_2^2 + (1+\|u\|_2^2)\|v_y\|_2^2\right)\,Y(t),\end{aligned}$$

where C is a constant. As a special consequence of this differential inequality, we conclude that, if

$$\int_0^T \|v(t)\|_{L^\infty}^2\, dt < \infty,$$

then $Y(t) < +\infty$ on $[0,T]$.

Proposition 1.8.2. *Let $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)$ and let (u, v, θ) be the corresponding classical solution of (1.29). Then,*

$$\sup_{r\ge 2} \frac{\|v(t)\|_{L^{2r}}}{\sqrt{r\log r}} \le \sup_{r\ge 2} \frac{\|v_0\|_{L^{2r}}}{\sqrt{r\log r}} + B(t), \tag{1.35}$$

where $B(t)$ is an explicit integrable function of $t \in [0,\infty)$ that depends on ν, κ and the initial norm $\|(u_0, v_0, \theta_0)\|_{H^2}$.

The proof of this proposition is very delicate and provided in another subsection. An exponential bound in r was obtained in [2] and a linear bound in [3], but we really need a bound at this growth level in order for the proof of Theorem 1.8.1 to work.

Proposition 1.8.3. *Let $s > 1$ and $f \in H^s(\mathbb{R}^2)$. Assume that*

$$\sup_{r\ge 2} \frac{\|f\|_r}{\sqrt{r\log r}} < \infty.$$

Then there exists a constant C depending on s only such that

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C \sup_{r\ge 2} \frac{\|f\|_r}{\sqrt{r\log r}} \\ \left[\log(e + \|f\|_{H^s(\mathbb{R}^2)}) \log\log(e + \|f\|_{H^s(\mathbb{R}^2)})\right]^{\frac{1}{2}}.$$

Proof of Theorem 1.8.1: Applying Proposition 1.8.2 and using the simple fact that $\|v\|_{H^2}^2 \le \|\omega\|_{H^1}^2 \le Y(t)$, we obtain

$$\frac{d}{dt} Y(t) \le A(t)Y(t) + C\, B^2(t)\, Y(t)\, \log(e + Y(t))\, \log\log(e + Y(t)),$$

where $A(t) = C\,(1 + \|\theta_0\|_\infty^2 + \|u_y\|_2^2 + (1 + \|u\|_2^2)\|v_y\|_2^2)$. An application of Gronwall's inequality then concludes the proof of Theorem 1.8.1.

1.8.3 Proof of Proposition 1.8.2

Before we provide the real proof, we would like to understand how a bound of this level can be obtained. For this purpose, we make the ansatz

$$\int_0^T \|p\|_\infty^2\, dt < \infty.$$

Then we show that

$$\|v\|_{L^{2r}} \leq C\sqrt{}$$

We also need two lemmas.

Lemma 1.8.2. *Let $f \in H^1(\mathbb{R}^2)$. Let $R > 0$. Denote by $B(0,R)$ the ball centered at zero with radius R and by $\chi_{B(0;R)}$ the characteristic function on $B(0,R)$. Write $f = \overline{f} + \widetilde{f}$ with*

$$\overline{f} = \mathcal{F}^{-1}(\chi_{B(0;R)}\mathcal{F}f) \quad \text{and} \quad \widetilde{f} = \mathcal{F}^{-1}((1-\chi_{B(0;R)})\mathcal{F}f). \tag{1.39}$$

Then we have the following estimates for $\overline{f}$ and $\widetilde{f}$.

(1) There exists a pure constant C independent of f and R such that

$$\|\overline{f}\|_{L^\infty(\mathbb{R}^2)} \leq C\sqrt{\log R}\,\|f\|_{H^1(\mathbb{R}^2)}. \tag{1.40}$$

(2) For any $2 \leq q < \infty$, there is a constant independent of q, R and f such that

$$\|\widetilde{f}\|_{L^q(\mathbb{R}^2)} \leq C\,\frac{q}{R^{\frac{2}{q}}}\,\|\widetilde{f}\|_{H^1(\mathbb{R}^2)} \leq C\,\frac{q}{R^{\frac{2}{q}}}\,\|f\|_{H^1(\mathbb{R}^2)} \tag{1.41}$$

In particular, for $q = 4$,

$$\|\widetilde{f}\|_{L^4(\mathbb{R}^2)} \leq \frac{C}{\sqrt{R}}\,\|f\|_{H^1(\mathbb{R}^2)}.$$

Lemma 1.8.3. *Let $q \in [2,\infty)$. Assume that $f, g, g_y, h_x \in L^2(\mathbb{R}^2)$ and $h \in L^{2(q-1)}(\mathbb{R}^2)$. Then*

$$\iint_{\mathbb{R}^2} |f\,g\,h|\,dxdy \leq C\,\|f\|_2\,\|g\|_2^{1-\frac{1}{q}}\|g_y\|_2^{\frac{1}{q}}\,\|h\|_{2(q-1)}^{1-\frac{1}{q}}\|h_x\|_2^{\frac{1}{q}}. \tag{1.42}$$

where C is a constant depending on q only. Two special cases of (1.42) are

$$\iint |f\,g\,h|\,dxdy \leq C\,\|f\|_2\,\|g\|_2^{\frac{2}{3}}\,\|g_y\|_2^{\frac{1}{3}}\,\|h\|_4^{\frac{2}{3}}\,\|h_x\|_2^{\frac{1}{3}} \tag{1.43}$$

and

$$\iint |f\,g\,h|\,dxdy \leq C\,\|f\|_2\,\|g\|_2^{\frac{1}{2}}\,\|g_y\|_2^{\frac{1}{2}}\,\|h\|_2^{\frac{1}{2}}\,\|h_x\|_2^{\frac{1}{2}}. \tag{1.44}$$

Proof of Proposition 1.8.2.

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \nu\,u_{yy}, \\ v_t + uv_x + vv_y = -p_y + \nu\,v_{yy} + \theta \end{cases} \tag{1.45}$$

Taking the inner product of the second equation in (1.45) with $v\,|v|^{2r-2}$ and integrating by parts, we obtain

$$\frac{1}{2r}\frac{d}{dt}\int |v|^{2r} \;+\; \nu(2r-1)\int v_y^2\,|v|^{2r-2}$$
$$= (2r-1)\int p\,v_y\;|v|^{2r-2} + \int \theta\,v\,|v|^{2r-2}$$
$$= (2r-1)\int \overline{p}\,v_y\;|v|^{2r-2} + (2r-1)\int \tilde{p}\,v_y\;|v|^{2r-2} + \int \theta\,v\,|v|^{2r-2}.$$

By Hölder's inequality,

$$\int \theta\,v\,|v|^{2r-2} \le \|\theta\|_{2r}\,\|v\|_{2r}^{2r-1}, \tag{1.46}$$
$$\int \overline{p}\,v_y\;|v|^{2r-2} \le \|\overline{p}\|_\infty\,\|v^{r-1}\|_2\,\|v_y v^{r-1}\|_2.$$

Applying Lemma 1.8.3, we have

$$\int \tilde{p}\,v_y\;|v|^{2r-2} \le C\|\tilde{p}\|_4^{\frac{2}{3}}\,\|\tilde{p}_x\|_2^{\frac{1}{3}}\,\|v^{r-1}\|_2^{\frac{2}{3}}\,\|(r-1)v_y v^{r-2}\|_2^{\frac{1}{3}}\,\|v_y v^{r-1}\|_2.$$

Furthermore, by Hölder's inequality,

$$\big\||v|^{r-1}\big\|_2 = \|v\|_{2(r-1)}^{r-1} \le \|v\|_2^{\frac{1}{r-1}}\,\|v\|_{2r}^{\frac{r(r-2)}{r-1}},$$
$$\big\||v|^{r-2}v_y\big\|_2^2 = \int |v|^{2(r-2)}v_y^2 = \int |v|^{2(r-2)}v_y^{\frac{2(r-2)}{r-1}}\,v_y^{\frac{2}{r-1}} \le \|v_y\|_2^{\frac{2}{r-1}}\,\big\|v_y\,|v|^{r-1}\big\|_2^{\frac{2(r-2)}{r-1}}.$$

Therefore,

$$\int \overline{p}\,v_y\;|v|^{2r-2} \;\le\; C\,\|\overline{p}\|_\infty\,\|v\|_2^{\frac{1}{r-1}}\,\|v\|_{2r}^{\frac{r(r-2)}{r-1}}\;\|v_y v^{r-1}\|_2,$$
$$\int \tilde{p}\,v_y\;|v|^{2r-2} \;\le\; C\,(r-1)^{\frac{1}{3}}\,\|\tilde{p}\|_4^{\frac{2}{3}}\,\|\tilde{p}_x\|_2^{\frac{1}{3}}\,\|v\|_2^{\frac{2}{3(r-1)}}\,\|v\|_{2r}^{\frac{2r(r-2)}{3(r-1)}}$$
$$\times\|v_y\|_2^{\frac{1}{3(r-1)}}\,\big\|v_y\,|v|^{r-1}\big\|_2^{1+\frac{(r-2)}{3(r-1)}}.$$

By Young's inequality and Lemma 1.8.2,

$$(2r-1)\int \overline{p}\,v_y\;|v|^{2r-2} \le \frac{\nu}{-}$$

Without loss of generality, we assume $\|v\|_{2r} \geq 1$. Inserting (1.46),(1.47) and (1.47) in (1.46), we have

$$\begin{aligned}
&\frac{1}{2r}\frac{d}{dt}\|v\|_{L^{2r}}^{2r} + \frac{\nu}{2}(2r-1)\int v_y^2|v|^{2r-2}\,dx \\
&\quad \leq C(2r-1)(\log R)\,\|p\|_{H^1}^2\|v\|_2^{\frac{2}{r-1}}\,\|v\|_{2r}^{2r-2} \\
&\quad +C\,(2r-1)(r-1)^{\frac{2r-2}{2r-1}}R^{-\frac{r-1}{2r-1}}\|p\|_{L^4}^{\frac{2r-2}{2r-1}}\|p\|_{H^1}^{\frac{4r-4}{2r-1}}\|v_y\|_2^{\frac{2}{2r-1}}\|v\|_2^{\frac{4}{2r-1}}\,\|v\|_{2r}^{2r-2} \\
&\quad +\|\theta\|_{L^{2r}}\,\|v\|_{L^{2r}}^{2r-1}.
\end{aligned}$$

Especially,

$$\begin{aligned}
\frac{d}{dt}\|v\|_{L^{2r}}^2 &\leq C(2r-1)(\log R)\,\|p\|_{H^1}^2\|v\|_2^{\frac{2}{r-1}} \\
&\quad + C\,(2r-1)(r-1)^{\frac{2r-2}{2r-1}}R^{-\frac{r-1}{2r-1}}\|p\|_{L^4}^{\frac{2r-2}{2r-1}}\left(\|p\|_{H^1}^2+\|v_y\|_2^2\right)\|v\|_2^{\frac{4}{2r-1}} \\
&\quad + \|\theta\|_{L^{2r}}^2 + \|v\|_{L^{2r}}^2.
\end{aligned}$$

Taking $R = (2r-1)^{\frac{2r-1}{2r-2}}(r-1)^2$, integrating in time and applying Propositions, we obtain

$$\|v(t)\|_{L^{2r}}^2 \leq \|v_0\|_{L^{2r}}^2 + B_1(t)r\log r + B_2(t),$$

where B_1 and B_2 are explicit integrable functions. Therefore,

$$\sup_{r\geq 2}\frac{\|v(t)\|_{L^{2r}}^2}{r\log r} \leq \sup_{r\geq 2}\frac{\|v_0\|_{L^{2r}}^2}{r\log r} + (B_1(t)+B_2(t)).$$

This completes the proof of Proposition 1.8.2. □

1.8.4 Proof of Proposition 1.8.3

This subsection proves Proposition 1.8.3.

Proof of Proposition 1.8.3. By the Littlewood-Paley decomposition, we can write

$$f = S_{N+1}f + \sum_{j=N+1}^{\infty}\Delta_j f,$$

where Δ_j denotes the Fourier localization operator and

$$S_{N+1} = \sum_{j=-1}^{N}\Delta_j.$$

The definitions of Δ_j and S_N are now standard. Therefore,

$$\|f\|_\infty \leq \|S_{N+1}f\|_\infty + \sum_{j=N+1}^{\infty}\|\Delta_j f\|_\infty.$$

We denote the terms on the right by I and II. By Bernstein's inequality, for any $q \geq 2$,

$$|I| \leq 2^{\frac{2N}{q}} \|S_{N+1} f\|_q \leq 2^{\frac{2N}{q}} \|f\|_q.$$

Taking $q = N$, we have

$$|I| \leq 4\|f\|_N \leq 4\sqrt{N \log N} \sup_{r \geq 2} \frac{\|f\|_r}{\sqrt{r \log r}}.$$

By Bernstein's inequality again, for any $s > 1$,

$$\begin{aligned} |II| \quad &\leq \quad \sum_{j=N+1}^{\infty} 2^j \|\Delta_j f\|_2 = \sum_{j=N+1}^{\infty} 2^{-j(s-1)} 2^{sj} \|\Delta_j f\|_2 \\ &= \quad C\, 2^{-(N+1)(s-1)} \|f\|_{B^s_{2,2}}. \end{aligned}$$

where C is a constant depending on s only. By identifying $B^s_{2;2}$ with H^s, we obtain

$$\|f\|_\infty \leq 4\sqrt{N \log N} \sup_{r \geq 2} \frac{\|f\|_r}{\sqrt{r \log r}} + C\, 2^{-(N+1)(s-1)} \|f\|_{H^s}.$$

We obtain the desired inequality (1.36) by taking

$$N = \left[\frac{1}{s-1} \log_2(e + \|f\|_{H^s}) \right],$$

where $[a]$ denotes the largest integer less than or equal to a. □

1.8.5 Open problems

It is currently not clear if the global regularity of Theorem 1.8.1 still holds if we set either $\nu = 0$ or $\kappa = 0$ in (1.29), even though we can verify that certain parts of the estimates still hold.

1.9 2D Boussinesq with fractional dissipation

This section is devoted to the global regularity of the 2D Boussinesq equations with fractional dissipation or fractional thermal diffusion. It is divided into two subsections.

1.9.1 Summary of several existing results

This subsection briefly summarizes some of the existing results on the global regularity issue concerning the 2D Boussinesq equations with fractional dissipation

$$(1.47) \qquad \begin{cases} \overrightarrow{u}_t + \overrightarrow{u} \cdot \nabla \overrightarrow{u} + \nu \Lambda \;\; \overrightarrow{u} = -\nabla p + \theta \overrightarrow{k}, \\ \nabla \cdot \overrightarrow{u} = 0, \\ \theta_t + \overrightarrow{u} \cdot \nabla \theta + \kappa \Lambda \;\; \theta = 0 \end{cases}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$. When $\alpha = \beta = 2$, these equations reduce to the standard 2D Boussinesq equations.

Hmidi, Keraani and Rousset [31] were able to establish the global regularity of the critical Navier-Stokes-Boussinesq equations, namely (1.47) with $\nu > 0$, $\kappa = 0$ and $\alpha = 1$. Their main result can be stated as follows.

Theorem 1.9.1. *Let $u_0 \in H^1 \cap \dot{W}^{1;q}$ with $q \in [2, \infty)$. Let $\theta_0 \in L^2 \cap B^0_{\infty;1}$. Then the Boussinesq equations have a unique global solution satisfying*

$$u \in L^\infty_{loc}([0,\infty); H^1 \cap \dot{W}^{1;q}) \cap L^1_{loc}([0,\infty); B^1_{\infty;1}),$$

$$\theta \in L^\infty_{loc}([0,\infty); L^2 \cap B^0_{\infty;1}).$$

We remark that direct energy estimates will not allow one to obtain even global L^2 bound on the vorticity. Their idea is to consider the evolution of the new function $\omega - \mathcal{R}\theta$. The vorticity equation is given by

$$\omega_t + u \cdot \nabla\omega + \Lambda\omega = \theta_{x_1}, \qquad \Lambda = (-\Delta)^{\frac{1}{2}}.$$

The idea is to write $\Lambda\omega - \theta_{x_1} = \Lambda(\omega - \mathcal{R}\theta)$, $\mathcal{R} = \Lambda^{-1}\partial_{x_1}$ and consider the difference with $(\mathcal{R}\theta)_t + \overrightarrow{u} \cdot \nabla(\mathcal{R}\theta) = -[\mathcal{R}, u \cdot \nabla]\theta$,

$$(\omega - \mathcal{R}\theta)_t + u \cdot \nabla(\omega - \mathcal{R}\theta) + \nu\Lambda(\omega - \mathcal{R}\theta) = [\mathcal{R}, u \cdot \nabla]\theta.$$

The advantage of this new equation is that the commutator is much more regular and thus can be controlled. In fact, the L^q-norm of this commutator is more or less bound by $\|\nabla u\|_{L^q} \|\theta\|_{B^0_{\infty,1}}$ and thus this new formation makes the global L^q bound for ω possible.

In addition, Hmidi, Keraani and Rousset [32] were also able to establish the global regularity of the critical Euler-Boussinesq equations, namely (1.47) with $\nu = 0$, $\kappa > 0$ and $\beta = 1$. Their main result can be stated as follows.

Theorem 1.9.2. *Let $u_0 \in B^1_{\infty;1} \cap \dot{W}^{1;q}$ with $q \in [2, \infty)$. Let $\theta_0 \in L^q \cap B^0_{\infty;1}$. Then the Boussinesq equations have a unique global solution satisfying*

$$u \in L^\infty_{loc}([0,\infty); B^1_{\infty;1} \cap \dot{W}^{1;q}),$$

$$\theta \in L^\infty_{loc}([0,\infty); L^q \cap B^0_{\infty;1}) \cap \tilde{L}^1_{loc}([0,\infty); B^1_{q;\infty}).$$

One key ingredient is to combine the vorticity equation

$$\omega_t + u \cdot \nabla\omega = \theta_{x_1}$$

with the equation for θ,

$$(\mathcal{R}\theta)_t + u \cdot \nabla(\mathcal{R}\theta) = -\mathcal{R}\Lambda\theta - [\mathcal{R}, u \cdot \nabla]\theta$$

into the equation

$$(\omega + \mathcal{R}\theta)_t + u \cdot \nabla(\omega + \mathcal{R}\theta) = -[\mathcal{R}, u \cdot \nabla]\theta.$$

Miao and Xue [42] studied the global regularity of (1.47) with $\nu > 0$ and $\kappa > 0$. Their result can be stated as follows.

Theorem 1.9.3. *Let* $\alpha \in ((6-\sqrt{6})/8, 1/2)$ *and*

$$\beta \in (1-\alpha)/2, \min\left((7+2\sqrt{6})\alpha/10 - 1, \alpha(1-\alpha)/2(\sqrt{6}-2\alpha), 1-\alpha\right).$$

Let $u_0 \in H^1 \cap W^{1;q}$ *with* $q \in (1/(\alpha+\beta-1), \infty)$ *and* $\theta_0 \in H^{1- \quad =2} \cap B^{1- \quad =2}_{\infty;1}$. *Then the fractional Boussinesq equations have a unique global solution.*

Finally we mention a rather recent global regularity result of P. Constantin and V. Vicol [20].

Theorem 1.9.4. *Consider the Boussinesq equations (1.47) with* $\nu > 0$ *and* $\kappa > 0$. *Assume that* $(u_0, \theta_0) \in \mathcal{S}$, *the Schwartz class. If* $\beta > \frac{2}{2+}$, *then (1.47) has a unique global smooth solution.*

1.9.2 The 2D Boussinesq equation with singular velocity

As a comparison, we first recall that the standard 2D Boussinesq equations,

$$\begin{cases} \overrightarrow{u}_t + \overrightarrow{u} \cdot \nabla \overrightarrow{u} + \Lambda \quad u = -\nabla p + \theta \overrightarrow{k}, \\ \nabla \cdot \overrightarrow{u} = 0, \\ \theta_t + \overrightarrow{u} \cdot \nabla \theta = 0 \end{cases}$$

with the vorticity $\omega = \nabla \times u$ satisfying

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \Lambda \quad \omega = \theta_{x_1}, \\ u = \nabla^\perp \psi \equiv (-\partial_{x_2}, \partial_{x_1})\psi, \quad \Delta \psi = \omega, \\ \partial_t \theta + u \cdot \nabla \theta = 0. \end{cases} \tag{1.48}$$

In a recent work in collaboration with Chae [16], we studied the following generalized vorticity formulation with a more singular velocity

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \Lambda \omega = \theta_{x_1}, \\ u = \nabla^\perp \psi \equiv (-\partial_{x_2}, \partial_{x_1})\psi, \quad \Delta \psi = \Lambda \quad (\log(I-\Delta)) \quad \omega, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \end{cases} \tag{1.49}$$

where $\sigma \geq 0$ and $\gamma \geq 0$, $\omega = \omega(x,t)$, $\psi = \psi(x,t)$ and $\theta = \theta(x,t)$ are scalar functions of $x \in \mathbb{R}^2$ and $t \geq 0$ while $u = u(x,t) : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field, and $\Lambda = (-\Delta)^{\frac{1}{2}}$ and Λ are Fourier multiplier operators with $\widehat{\Lambda \quad f}(\xi) = |\xi| \quad \widehat{f}(\xi)$ and

$$\mathcal{F}((\log(I-\Delta)) \quad f)(\xi) = (\log(I+|\xi|^2)) \quad \widehat{f}(\xi).$$

In the case when $\sigma = 0$ and $\gamma = 0$, (1.49) reduces to the standard vorticity formulation of the 2D Boussinesq vorticity equation (1.48). When either $\sigma > 0$ or $\gamma > 0$, the velocity in (1.49) is more singular.

Our motivation for studying the global regularity of (1.49) comes from two different sources: the first being the models generalizing the surface quasi-geostrophic equation and the 2D hydrodynamics equations (see, e.g. the papers of Constantin, Iyer and Wu [19], of Kiselev [35], of Chae, Constantin and Wu [11], and of Dabkowski, Kiselev and Vicol [21], etc) and the second being the Boussinesq-Navier-Stokes system with critical dissipation (see, e.g. Hmidi, Keraani and Rousset [31, 32]). Our goal here is to extend the work of Hmidi, Keraani and Rousset [31, 32] to cover more singular velocities and explore how far one can go beyond the critical case.

(1.49) does have a corresponding velocity formulation. In fact, v satisfying $u = \Lambda\ (\log(I-\Delta))\ v$ plays the role of the standard velocity.

Theorem 1.9.5. *For classical solutions of* (1.49) *that decay sufficiently fast as* $|x| \to \infty$, (1.49) *is equivalent to the following equations*

$$\begin{cases} \partial_t v + u\cdot\nabla v - \sum_{j=1}^{2} u_j \nabla v_j + \nu\Lambda\ v = -\nabla p + \theta \mathbf{e}_2, \\ \nabla\cdot v = 0, \quad u = \Lambda\ (\log(I-\Delta))\ v, \\ \partial_t\theta + u\cdot\nabla\theta = 0. \end{cases} \tag{1.50}$$

Our main result concerning the global regularity of solutions to (1.49) can be stated as follows.

Theorem 1.9.6. *Consider the generalized Boussinesq equations (1.49) with* $\sigma = 0$ *and* $\gamma \geq 0$. *Assume the initial data* (ω_0, θ_0) *satisfies*

$$\omega_0 \in L^2 \cap L^q \cap B^{0;}_{\infty;1}, \quad \theta_0 \in L^2 \cap B^{0;}_{\infty;1}$$

for some $q > 2$. *Then (1.49) has a unique global solution* (ω, θ) *satisfying, for any* $t > 0$,

$$\omega \in L^2 \cap L^q \cap L^1_t B^{0;}_{\infty;1}, \quad \theta \in L^2 \cap L^\infty \cap L^1_t B^{0;}_{\infty;1}.$$

We remark that this is a global regularity result for the logarithmically supercritical case. Although it is not clear if this global regularity result still holds for the more singular case when $\sigma > 0$, we can still show that the L^2-norm of the vorticity ω is bounded at any time for $0 \leq \sigma < \frac{1}{2}$ and $\gamma \geq 0$. More precisely, we have the following theorem.

Theorem 1.9.7. *Consider (1.49) with* $0 \leq \sigma < \frac{1}{2}$ *and* $\gamma \geq 0$. *Assume* (ω_0, θ_0) *satisfies the conditions stated in previous Theorem. Let* (ω, θ) *be the corresponding solution. Then, for any* $t > 0$,

$$\|\omega(t)\|_{L^2} \leq B(t)$$

for a smooth function $B(t)$ *of* t *depending on the initial data only. In addition,* $G = \omega - \mathcal{R}\theta$ *satisfies the basic energy bound*

$$\|G(t)\|^2_{L^2} + \int_0^t \|\Lambda^{\frac{1}{2}} G\|^2_{L^2} dt \leq B(t). \tag{1.51}$$

We explain how to prove Theorem (1.9.6). There are three key ingredients in the proof. The first is to consider the combined quantity $G = \omega - \mathcal{R}\theta$, which satisfies

$$\partial_t G + u \cdot \nabla G + \Lambda G = [\mathcal{R}, u \cdot \nabla]\theta. \tag{1.52}$$

obtained by taking the difference

$$\omega_t + u \cdot \nabla\omega + \Lambda(\omega - \mathcal{R}\theta) = 0$$

and

$$(\mathcal{R}\theta)_t + \overrightarrow{u} \cdot \nabla(\mathcal{R}\theta) = -[\mathcal{R}, u \cdot \nabla]\theta$$

The second is the following commutator estimate.

Proposition 1.9.1. *Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field. Let $\mathcal{R} = \partial_{x_1}\Lambda^{-1}$ denote a Riesz transform. Let $s \in (0,1)$, $s < \delta < 1$, $p \in (1,\infty)$ and $q \in [1,\infty]$. Then*

$$\|[\mathcal{R}, u]F\|_{B^s_{p,q}} \leq C_1 \|u\|_{\mathring{B}^\delta_{p,\infty}} \|F\|_{B^{s-\delta}_{\infty,q}} + C_2\|u\|_{L^2} \|F\|_{L^2}, \tag{1.53}$$

When $\delta = 1$, $\|u\|_{\mathring{B}^\delta_{p,\infty}}$ is replaced by $\|\nabla u\|_{L^p}$.

A special consequence of Proposition 1.9.1 is an estimate for u related to ω as in (1.49).

Corollary 1.9.1. *Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field determined by a scalar function ω through the relations*

$$u = \nabla^\perp\psi, \quad \Delta\psi = \Lambda \ \ (\log(I - \Delta)) \ \ \omega, \tag{1.54}$$

where $0 \leq \sigma < \frac{1}{2}$ and $\gamma \geq 0$ are real parameters. Then, for any $0 \leq s < 1-\sigma$, $p \in (1,\infty)$ and $q \in [1,\infty]$,

$$\|[\mathcal{R}, u]\theta\|_{B^s_{p,q}} \leq C \|\omega\|_{L^p}\|\theta\|_{B^{s+\sigma-1}_{\infty,q}} + C \|u\|_{L^2} \|\theta\|_{L^2}, \tag{1.55}$$

The proof of Proposition 1.9.1 makes use of the following commutator estimate.

Lemma 1.9.1. *Let $p \in [1,\infty]$ and $\delta \in (0,1)$. If $|x| \ \phi \in L^1$, $f \in \mathring{B}_{p;\infty}$ and $g \in L^\infty$, then*

$$\|\phi * (fg) - f(\phi * g)\|_{L^p} \leq C\||x| \ \phi\|_{L^1}\|f\|_{\mathring{B}^\delta_{p,\infty}}\|g\|_{L^\infty}. \tag{1.56}$$

In the case when $\delta = 1$, (1.56) is replaced by

$$\|\phi * (fg) - f(\phi * g)\|_{L^p} \leq C\||x|\phi\|_{L^1}\|\nabla f\|_{L^p}\|g\|_{L^\infty}. \tag{1.57}$$

The third ingredient is to prove $\omega \in L^q$ when $\omega_0 \in L^q$ with $q \in [2,\infty)$ through two steps. The first step is

Theorem 1.9.8. *Consider (1.49) with* $0 \le \sigma < \frac{1}{2}$ *and* $\gamma \ge 0$. *Assume that* (ω_0, θ_0) *satisfies the conditions in Theorem 1.9.6, especially* $(\omega_0, \theta_0) \in L^q$ *for* $q \in (2, \frac{4}{2\ +1}]$. *Let* (ω, θ) *be the corresponding solution of (1.49). Then, for* $q \in (2, \frac{4}{2\ +1})$ *with* $\gamma > 0$ *and* $q \in (2, \frac{4}{2\ +1}]$ *with* $\gamma = 0$, *and any* $t > 0$,

$$\|\omega(t)\|_{L^q} \le B(t), \tag{1.58}$$

$$\|G(t)\|^q_{L^q} + C\int_0^t \|G(\tau)\|^q_{L^{2q}}\, d\tau \le B(t), \tag{1.59}$$

where $B(t)$*'s are smooth functions of* t.

The second step is a proposition showing that $\|G\|_{L^1_t B^s_{q,1}} \le B(t)$.

Theorem 1.9.9. *Consider (1.49) with* $0 \le \sigma < \frac{1}{4}$ *and* $\gamma \ge 0$. *Assume that* (ω_0, θ_0) *satisfies the conditions in Theorem 1.9.6. Let* (ω, θ) *be the corresponding solution of (1.49). Let* r, q *and* s *satisfy*

$$r \in [1, \infty], \quad s < 1 - \sigma, \quad \frac{2}{1-\sigma} < q < \frac{4}{1+2\sigma}.$$

In the case when $\gamma = 0$, *we can take* $q = 4/(1 + 2\sigma)$. *Then, for any* $t > 0$,

$$\|G\|_{\tilde{L}^r_t B^s_{q,1}} \le B(t). \tag{1.60}$$

Theorem 1.9.10. *Consider (1.49) with* $\sigma = 0$ *and* $\gamma \ge 0$. *Assume that* (ω_0, θ_0) *satisfies the conditions in Theorem 1.9.6, especially* $(\omega_0, \theta_0) \in B^{0;}_{\infty;1}$. *Let* (ω, θ) *be the corresponding solution of (1.49). Then, for any* $t > 0$,

$$\|\omega\|_{L^1_t B^{0,\gamma}_{\infty,1}} \le B(t), \quad \|\theta\|_{L^1_t B^{0,\gamma}_{\infty,1}} \le B(t). \tag{1.61}$$

Proof of Theorem 1.9.10. $\|G\|_{L^1_t B^s_{q,1}} \le B(t)$ especially implies $\|G\|_{L^1_t B^{0,\gamma}_{\infty,1}} \le B(t)$. Since $G = \omega - R\theta$,

$$\|\omega\|_{B^{0,\gamma}_{\infty,1}} \le \|G\|_{B^{0,\gamma}_{\infty,1}} + \|\mathcal{R}\theta\|_{B^{0,\gamma}_{\infty,1}}.$$

In addition,

$$\|\mathcal{R}\theta\|_{B^{0,\gamma}_{\infty,1}} \le \|\Delta_{-1}\theta\|_{L^\infty} + \|\theta\|_{B^{0,\gamma}_{\infty,1}} \le \|\theta_0\|_{L^2} + \|\theta\|_{B^{0,\gamma}_{\infty,1}}$$

According to Lemma 1.9.2 below, we have

$$\|\nabla u\|_{L^\infty} \le \|\omega\|_{L^2} + \|\omega\|_{B^{0,\gamma}_{\infty,1}},$$

$$\|\theta\|_{B^{0,\gamma}_{\infty,1}} \le \|\theta_0\|_{B^{0,\gamma}_{\infty,1}} \left(1 + \int_0^t \|\omega\|_{L^2}\, dt\right) + \|\theta_0\|_{B^{0,\gamma}_{\infty,1}} \int_0^t \|\omega\|_{B^{0,\gamma}_{\infty,1}}\, dt.$$

$$\begin{aligned}\|\omega\|_{B^{0,\gamma}_{\infty,1}} \le\ & \|G\|_{B^{0,\gamma}_{\infty,1}} + \|\theta_0\|_{L^2} + \|\theta_0\|_{B^{0,\gamma}_{\infty,1}} \left(1 + \int_0^t \|\omega\|_{L^2}\, dt\right) \\ & + \|\theta_0\|_{B^{0,\gamma}_{\infty,1}} \int_0^t \|\omega\|_{B^{0,\gamma}_{\infty,1}}\, dt.\end{aligned}$$

□

Lemma 1.9.2. *Let θ satisfy*

$$\partial_t \theta + u \cdot \nabla \theta = f.$$

Let $\gamma \geq 0$ and $\rho \in [1, \infty]$. Then, for any $t > 0$,

$$\|\theta(t)\|_{B^{0,\gamma}_{\rho,1}} \leq \left(\|\theta_0\|_{B^{0,\gamma}_{\rho,1}} + \|f\|_{L^1_t B^{0,\gamma}_{\rho,1}} \right) \left(1 + \int_0^t \|\nabla u\|_{L^\infty} \, dt \right).$$

Theorem 1.9.11. *Consider (1.49) with $\sigma = 0$ and $\gamma \geq 0$. Assume (ω_0, θ_0) satisfies the conditions stated in Theorem 1.9.6. Let (ω, θ) be the corresponding solution. Then, for any $q \geq 2$,*

$$\|\omega(t)\|_{L^q} \leq B(t). \tag{1.62}$$

Proof. It is clear from (1.52) that, for any $q \geq 2$,

$$\|G\|_{L^q} \leq \|G_0\|_{L^q} + \int_0^t \|[\mathcal{R}, u \cdot \nabla \theta]\|_{L^q} \, dt.$$

According to the commutator estimate of Proposition 1.9.2 below,

$$\|G\|_{L^q} \leq \|G_0\|_{L^q} + \int_0^t \|\omega(s)\|_{L^q} \, \|\theta(s)\|_{B^{0,\gamma}_{\infty,1}} \, ds.$$

Therefore,

$$\|\omega(t)\|_{L^q} \leq \|\theta_0\|_{L^q} + \|G_0\|_{L^q} + \int_0^t \|\omega(s)\|_{L^q} \, \|\theta(s)\|_{B^{0,\gamma}_{\infty,1}} \, ds.$$

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